Angular Position

1. Axis of rotation is the center of the disc.
2. Choose a fixed reference line.
3. Point $P$ is at a fixed distance $r$ from the origin.

Angular Position, 2

- Point $P$ will rotate about the origin in a circle of radius $r$.
- Every particle on the disc undergoes circular motion about the origin, $O$.
- $P$ is located at $(r, \theta)$ where $r$ is the distance from the origin to $P$ and $\theta$ is the measured counterclockwise from the reference line.

Angular Position, 3

- As the particle moves, the only coordinate that changes is $\theta$.
- As the particle moves through $\theta$, it moves through an arc length $s$.
- The arc length and $r$ are related:
  - $s = \theta r$.
Radian

- This can also be expressed as 
  \[ \theta = \frac{s}{r} \]
- \(\theta\) is a pure number, but commonly is given the artificial unit, radian
- **One radian is the angle subtended by an arc length equal to the radius of the arc**

Conversions

- Comparing degrees and radians
  \[ 1 \text{ rad} = \frac{360^\circ}{2\pi} \]
- Converting from degrees to radians
  \[ \theta[\text{rad}] = \frac{\pi}{180^\circ} \theta \text{ [degrees]} \]

Angular Displacement

- The **angular displacement** is defined as the angle the object rotates through during some time interval
  \[ \Delta \theta = \theta_f - \theta_i \]

Average Angular Speed

- The average angular speed, \(\bar{\omega}\), of a rotating rigid object is the ratio of the angular displacement to the time interval
  \[ \bar{\omega} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta \theta}{\Delta t} \]
Average Angular Acceleration

- The average angular acceleration, $\alpha$, of an object is defined as the ratio of the change in the angular speed to the time it takes for the object to undergo the change:

$$\alpha = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta \omega}{\Delta t}$$

Rotational Kinematic Equations

$$\begin{align*}
\omega_f &= \omega_i + \alpha t \\
\theta_f &= \theta_i + \omega_i t + \frac{1}{2} \alpha t^2 \\
\omega_f^2 &= \omega_i^2 + 2\alpha(\theta_f - \theta_i) \\
\theta_f &= \theta_i + \frac{1}{2}(\omega_i + \omega_f) t
\end{align*}$$

Comparison Between Rotational and Linear Equations

Table 10.1

<table>
<thead>
<tr>
<th>Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotational Motion</td>
</tr>
<tr>
<td>Rotational Motion About Fixed Axis</td>
</tr>
<tr>
<td>$\omega_f = \omega_i + \alpha t$</td>
</tr>
<tr>
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</tbody>
</table>

Relationship Between Angular and Linear Quantities

- Displacements
  $$s = \theta r$$
- Speeds
  $$v = \omega r$$
- Accelerations
  $$a = \alpha r$$
- Every point on the rotating object has the same angular motion
- Every point on the rotating object does not have the same linear motion
Speed Comparison

- The linear velocity is always tangent to the circular path
  - called the tangential velocity
- The magnitude is defined by the tangential speed
  \[ v = \frac{ds}{dt} = r \frac{d\theta}{dt} = r \omega \]

Acceleration Comparison

- The tangential acceleration is the derivative of the tangential velocity
  \[ a_t = \frac{dv}{dt} = r \frac{d\omega}{dt} = r \alpha \]

Centripetal Acceleration

- An object traveling in a circle, even though it moves with a constant speed, will have an acceleration
  - Therefore, each point on a rotating rigid object will experience a centripetal acceleration
  \[ a_c = \frac{v^2}{r} = \frac{(r \omega)^2}{r} = r \omega^2 \]

Rotational Motion Example

- For a compact disc player to read a CD, the angular speed must vary to keep the tangential speed constant \( (v_t = \omega r) \)
- At the inner sections, the angular speed is faster than at the outer sections
Rotational Kinetic Energy

- An object rotating about some axis with an angular speed, \( \omega \), has rotational kinetic energy even though it may not have any translational kinetic energy.
- Each particle has a kinetic energy of \( K_i = \frac{1}{2} m_i v_i^2 \).
- Since the tangential velocity depends on the distance, \( r \), from the axis of rotation, we can substitute \( v_i = \omega_i r \).

Rotational Kinetic Energy, cont

- The total rotational kinetic energy of the rigid object is the sum of the energies of all its particles:
  \[
  K_s = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega_i^2
  \]
  \[
  K_s = \frac{1}{2} \left( \sum_i m_i r_i^2 \right) \omega^2 = \frac{1}{2} I \omega^2
  \]
- Where \( I \) is called the moment of inertia.

Rotational Kinetic Energy, final

- There is an analogy between the kinetic energies associated with linear motion (\( K = \frac{1}{2} mv^2 \)) and the kinetic energy associated with rotational motion (\( K_{rot} = \frac{1}{2} I \omega^2 \)).
- Rotational kinetic energy is not a new type of energy, the form is different because it is applied to a rotating object.
- The units of rotational kinetic energy are Joules (J).

Moment of Inertia

- The definition of moment of inertia is
  \[
  I = \sum r_i^2 m_i
  \]
- The dimensions of moment of inertia are \( ML^2 \) and its SI units are \( kg \cdot m^2 \).
- We can calculate the moment of inertia of an object more easily by assuming it is divided into many small volume elements, each of mass \( \Delta m_i \).
Moments of Inertia of Various Rigid Objects

**Torque**

- Torque, \( \tau \), is the tendency of a force to rotate an object about some axis
  - Torque is a vector
  - \( \tau = r \sin \phi \cdot F \)
    - \( F \) is the force
    - \( \phi \) is the angle the force makes with the horizontal
    - \( r \) is the moment arm (or lever arm)

**Torque, cont**

- The moment arm, \( r \), is the perpendicular distance from the axis of rotation to a line drawn along the direction of the force
  - \( d = r \sin \Phi \)

**Torque, final**

- The horizontal component of \( F \) (\( F \cos \phi \)) has no tendency to produce a rotation
- Torque will have direction
  - If the turning tendency of the force is counterclockwise, the torque will be positive
  - If the turning tendency is clockwise, the torque will be negative
**Net Torque**

- The force $F_1$ will tend to cause a counterclockwise rotation about $O$.
- The force $F_2$ will tend to cause a clockwise rotation about $O$.
- $\Sigma r = r_1 + r_2 = F_1 d_1 - F_2 d_2$

**Torque and Angular Acceleration**

- Consider a particle of mass $m$ rotating in a circle of radius $r$ under the influence of tangential force $F_r$.
- The tangential force provides a tangential acceleration: $F_r = ma_t$.

---

**Torque and Angular Acceleration, Particle cont.**

- Since $mr^2$ is the moment of inertia of the particle,
  - $\tau = I \alpha$
  - The torque is directly proportional to the angular acceleration and the constant of proportionality is the moment of inertia.

---

**Summary of Useful Equations**

<table>
<thead>
<tr>
<th>Rotational Motion About a Fixed Axis</th>
<th>Linear Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular speed $\omega = d\theta/dt$</td>
<td>Linear speed $v = ds/dt$</td>
</tr>
<tr>
<td>Angular acceleration $a = d\omega/dt$</td>
<td>Linear acceleration $a = dv/dt$</td>
</tr>
<tr>
<td>Net torque $\Sigma r = I \alpha$</td>
<td>Net force $\Sigma F = ma$</td>
</tr>
<tr>
<td>$a = \text{constant}$ $\omega^2 = \omega_1^2 + 2a(\theta_1 - \theta)$</td>
<td>$a = \text{constant}$ $v^2 = v_0^2 + 2a(s_0 - s_0)$</td>
</tr>
<tr>
<td>Work $W = \int r , d\theta$</td>
<td>Work $W = \int F_x , dx$</td>
</tr>
<tr>
<td>Rotational kinetic energy $K = \frac{1}{2} I \omega^2$</td>
<td>Kinetic energy $K = \frac{1}{2}mv^2$</td>
</tr>
<tr>
<td>Power $P = \tau \omega$</td>
<td>Power $P = F_x v$</td>
</tr>
<tr>
<td>Angular momentum $L = I \omega$</td>
<td>Linear momentum $p = mv$</td>
</tr>
<tr>
<td>Net torque $\Sigma r = dL/dt$</td>
<td>Net force $\Sigma F = dp/dt$</td>
</tr>
</tbody>
</table>
Total Kinetic Energy of a Rolling Object

- The total kinetic energy of a rolling object is the sum of the translational energy of its center of mass and the rotational kinetic energy about its center of mass
  \[ K = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} MV_{CM}^2 \]

Total Kinetic Energy, Example

- Accelerated rolling motion is possible only if friction is present between the sphere and the incline
  - The friction produces the net torque required for rotation

Total Kinetic Energy, Example cont

- Despite the friction, no loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant
- Let \( U = 0 \) at the bottom of the plane
- \( K_f + U_f = K_i + U_i \)
- \( K_i = \frac{1}{2} (I_{CM} / R^2) v_{CM}^2 + \frac{1}{2} MV_{CM}^2 \)
- \( U_i = Mgh \)
- \( U_f = K_f = 0 \)