Chapter 11
Angular Momentum

The Vector Product

- There are instances where the product of two vectors is another vector
  - Earlier we saw where the product of two vectors was a scalar
    - This was called the dot product
  - The vector product of two vectors is also called the cross product

The Vector Product and Torque

- The torque vector lies in a direction perpendicular to the plane formed by the position vector and the force vector
  - \( \mathbf{\tau} = \mathbf{r} \times \mathbf{F} \)
- The torque is the vector (or cross) product of the position vector and the force vector

The Vector Product Defined

- Given two vectors, \( \mathbf{A} \) and \( \mathbf{B} \)
  - The vector (cross) product of \( \mathbf{A} \) and \( \mathbf{B} \) is defined as a third vector, \( \mathbf{C} \)
  - \( \mathbf{C} \) is read as “\( \mathbf{A} \) cross \( \mathbf{B} \)”
- The magnitude of \( \mathbf{C} \) is \( AB \sin \theta \)
  - \( \theta \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \)
More About the Vector Product

- The quantity $AB \sin \theta$ is equal to the area of the parallelogram formed by $A$ and $B$
- The direction of $C$ is perpendicular to the plane formed by $A$ and $B$
- The best way to determine this direction is to use the right-hand rule

Properties of the Vector Product

- The vector product is \textit{not} commutative. The order in which the vectors are multiplied is important
- To account for order, remember $A \times B = -B \times A$
- If $A$ is parallel to $B$ ($\theta = 0^\circ$ or $180^\circ$), then $A \times B = 0$
  - Therefore $A \times A = 0$

Vector Products of Unit Vectors

- $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$
- $\hat{i} \times \hat{j} = -\hat{j} \times \hat{i} = \hat{k}$
- $\hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i}$
- $\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$

Vector Products of Unit Vectors, cont

- Signs are interchangeable in cross products
  - $A \times (-B) = -A \times B$
  - $\hat{i} \times (-\hat{j}) = -\hat{i} \times \hat{j}$
Using Determinants

- The cross product can be expressed as
  \[
  \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}
  \]
- Expanding the determinants gives
  \[
  \mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y) \hat{i} - (A_x B_z - A_z B_x) \hat{j} + (A_x B_y - A_y B_x) \hat{k}
  \]

Torque Vector Example

- Given the force
  \[
  \mathbf{F} = (2.00 \hat{i} + 3.00 \hat{j}) \text{ N}
  \]
  \[
  \mathbf{r} = (4.00 \hat{i} + 5.00 \hat{j}) \text{ m}
  \]

  \[
  \tau = \mathbf{r} \times \mathbf{F} = [(4.00 \hat{i} + 5.00 \hat{j}) \text{ m}] \times [(2.00 \hat{i} + 3.00 \hat{j}) \text{ N}]
  \]
  \[
  = [(4.00)(2.00) \hat{i} \times \hat{i} + (4.00)(3.00) \hat{i} \times \hat{j}]
  \]
  \[
  + (5.00)(2.00) \hat{j} \times \hat{i} + (5.00)(3.00) \hat{j} \times \hat{j}
  \]
  \[
  = 2.00 \hat{k} \text{ N} \cdot \text{m}
  \]

Angular Momentum

- Consider a particle of mass \( m \) located at the vector position \( \mathbf{r} \) and moving with linear momentum \( \mathbf{p} \)

  \[
  \mathbf{r} \times \sum \mathbf{F} = \sum \tau = \mathbf{r} \times \frac{d\mathbf{p}}{dt}
  \]

  Adding the term \( \frac{d\mathbf{r}}{dt} \times \mathbf{p} \)

  \[
  \sum \tau = \frac{d(r \times \mathbf{p})}{dt}
  \]

Angular Momentum, cont

- The instantaneous angular momentum \( \mathbf{L} \) of a particle relative to the origin \( O \) is defined as the cross product of the particle’s instantaneous position vector \( \mathbf{r} \) and its instantaneous linear momentum \( \mathbf{p} \)

  \[
  \mathbf{L} = \mathbf{r} \times \mathbf{p}
  \]
Torque and Angular Momentum

- The torque is related to the angular momentum
  - Similar to the way force is related to linear momentum
    \[ \sum \tau = \frac{d(r \times p)}{dt} \quad \sum \tau = \frac{dL}{dt} \]
- This is the rotational analog of Newton’s Second Law
  - \( \sum \tau \) and \( L \) must be measured about the same origin
  - This is valid for any origin fixed in an inertial frame

More About Angular Momentum

- The SI units of angular momentum are \( r \times p = (kg \cdot m^2)/s \)
- Both the magnitude and direction of \( L \) depend on the choice of origin
- The magnitude of \( L = mvr \sin \phi \)
  - \( \phi \) is the angle between \( p \) and \( r \)
- The direction of \( L \) is perpendicular to the plane formed by \( r \) and \( p \)

Angular Momentum of a Particle, Example

- The vector \( L = r \times p \) is pointed out of the diagram
- The magnitude is \( L = mvr \sin 90^\circ = mvr \)
  - \( \sin 90^\circ \) is used since \( v \) is perpendicular to \( r \)
- A particle in uniform circular motion has a constant angular momentum about an axis through the center of its path

Angular Momentum of a Rotating Rigid Object

- Each particle of the object rotates in the \( xy \) plane about the \( z \) axis with an angular speed of \( \omega \)
- The angular momentum of an individual particle is \( L_i = m_i r_i^2 \omega \)
- \( L \) and \( \omega \) are directed along the \( z \) axis
Angular Momentum of a Rotating Rigid Object, cont

- To find the angular momentum of the entire object, add the angular momenta of all the individual particles
  \[ L_i = \sum m_i r_i^2 \omega = I_i \omega \]
- This also gives the rotational form of Newton’s Second Law
  \[ \sum \tau = \sum I \frac{d\omega}{dt} = 1 \alpha \]

Angular Momentum of a Bowling Ball

- The momentum of inertia of the ball is \( 2/5 MR^2 \)
- The angular momentum of the ball is \( L_z = I \omega \)
- The direction of the angular momentum is in the positive \( z \) direction

Conservation of Angular Momentum

- The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero
  - Net torque = 0 \( \rightarrow \) means that the system is isolated
- \( L_{\text{tot}} = \text{constant} \) or \( L_f = L_i \)
- For a system of particles, \( L_{\text{tot}} = \sum L_i = \text{constant} \)

Conservation of Angular Momentum, cont

- If the mass of an isolated system undergoes redistribution, the moment of inertia changes
  - The conservation of angular momentum requires a compensating change in the angular velocity
    - \( I \omega_i = I \omega_f \)
    - This holds for rotation about a fixed axis and for rotation about an axis through the center of mass of a moving system
  - The net torque must be zero in any case
Conservation Law Summary

- For an isolated system -
  1. Conservation of Energy:
     - \( E_i = E_f \)
  2. Conservation of Linear Momentum:
     - \( p_i = p_f \)
  3. Conservation of Angular Momentum:
     - \( L_i = L_f \)

Conservation of Angular Momentum: The Merry-Go-Round

- The moment of inertia of the system is the moment of inertia of the platform plus the moment of inertia of the person
- Assume the person can be treated as a particle
- As the person moves toward the center of the rotating platform, the angular speed will increase
- To keep \( L \) constant

A playground merry-go-round of radius \( R = 1.40 \text{ m} \) has a moment of inertia \( I = 250 \text{ kg} \cdot \text{m}^2 \) and is rotating at 11.0 rev/min about a frictionless vertical axle. Facing the axle, a 25.0 kg child hops onto the merry-go-round and manages to sit down on its edge. What is the new angular speed of the merry-go-round? Answer: 9.2 rev/min