Answers to Selected Exercises for Chapter 7

Section 7.1 (page 544)

1. (a) explicit, two-step (b) explicit, one-step (c) implicit, two-step
   (d) implicit, one-step (e) explicit, two-step

3. (a) $|f(t,y_1) - f(t,y_2)| = |t||y_1 - y_2||y_1^2 + y_1y_2 + y_2^2| \le 300|y_1 - y_2|$
   (b) $|f(t,y_1) - f(t,y_2)| = |\frac{t}{1+t^2}||y_1 - y_2|\frac{1}{|\sqrt{y_1}+\sqrt{y_2}|} \le \frac{1}{4}|y_1 - y_2|$
   (c) $|f(t,y_1) - f(t,y_2)| = e^t|\frac{y_1-y_2}{y_1y_2}| \le e^2|y_1 - y_2|$
   (d) $|\partial f/\partial y| \le 10e^2 + 1$
   (e) $|\partial f/\partial y| \le 22$

5. $|f(t,y_1) - f(t,y_2)| = |t||y_1 - y_2||y_1 + y_2 + 1| \le (2M+1)|y_1 - y_2|$; the function does not satisfy a Lipschitz condition when y can be any real number.

7. $y_2(t) = -t - \frac{1}{3}t^3 + \frac{4}{15}t^5 - \frac{2}{63}t^7$

9. $y_4(t) = 1 - t + t^2 - \frac{1}{3}t^3 + \frac{1}{12}t^4 - \frac{1}{120}t^5$

11. $1 + t + t^2 + \frac{2}{3}t^3 + \frac{5}{6}t^4$

13. $1 + t + \frac{1}{6}t^3 - \frac{1}{8}t^4$

Section 7.2 (page 558)

1.

t_i	w_i
0.00	1.00000000
0.25	0.75000000
0.50	0.58886719
0.75	0.46717517
1.00	0.36949930

3.

t_i	w_i
0.00	0.00000000
0.33	−0.33333333
0.67	−0.94105313
1.00	−2.50087572

5.

t_i	w_i
0.00	0.00000000
0.25	−0.25000000
0.50	−0.51562500
0.75	−0.83605957
1.00	−1.29493237

7.

t_i	w_i	Error	Error Bound
0.00	1.00000000	0.000000	0.000000
0.50	1.50000000	0.015732	0.612552
1.00	2.04957376	0.056741	25.253406
1.50	2.71270721	0.109239	1016.470789
2.00	3.53876191	0.173126	40889.759259

9.

t_i	w_i	Error	Error Bound
0.00	1.00000000	0.000000	0.000000
1.00	0.00000000	0.735759	1.718282
2.00	1.00000000	0.270671	6.389056
3.00	2.00000000	0.099574	19.085537
4.00	3.00000000	0.036631	53.598150

11.

t_i	w_i	$\tilde{w}_i$	$\lvert w_i - \tilde{w}_i\rvert$	$e^{(t_i-a)L}\lvert\alpha - \tilde{\alpha}\rvert$
2.00	-5.00000000	-5.10000000	0.10000000	0.100000
2.40	-3.40000000	-3.46800000	0.06800000	0.137713
2.80	-2.43431953	-2.48300592	0.04868639	0.189648
3.20	-1.81747838	-1.85382795	0.03634957	0.261170
3.60	-1.40353313	-1.43160379	0.02807066	0.359664
4.00	-1.11397901	-1.13625859	0.02227958	0.495303

13.

t_i	w_i	$\tilde{w}_i$	$\lvert w_i - \tilde{w}_i\rvert$	$e^{(t_i-a)L}\lvert\alpha - \tilde{\alpha}\rvert$
0.00	1.00000000	0.90000000	0.10000000	0.013534
1.00	2.00000000	2.00000000	0.00000000	0.036788
2.00	5.43656366	5.43656366	0.00000000	0.100000
3.00	14.77811220	14.77811220	0.00000000	0.271828
4.00	40.17107385	40.17107385	0.00000000	0.738906
5.00	109.19630007	109.19630007	0.00000000	2.008554

17. 2

19.

h	$w_h(2)$	Error	Ratio
1/8	47.01489874	27.717490	0.163213
1/16	58.72429222	16.008096	1.731467
1/32	66.06076195	8.671626	1.846032
1/64	70.20852189	4.523866	1.916862
1/128	72.42039078	2.311998	1.956692
1/256	73.56346244	1.168926	1.977882
1/512	74.14464021	0.587748	1.988821

21.

h	$w_h(2)$	$\frac{w_h(2)-w_{h/2}(2)}{w_{h/2}(2)-w_{h/4}(2)}$
1/8	-1.02147625	2.738164
1/16	-1.03698250	2.260992
1/32	-1.04264552	2.114309
1/64	-1.04515017	2.053844
1/128	-1.04633480	2.026164
1/256	-1.04691158	
1/512	-1.04719625	

23. (a) Rate of convergence is $O(h)$
(b) To within roundoff error, the Euler's method approximation is exact for any step size
(c) The error term for Euler's method involves the second derivative of the solution; in part (b), the second derivative of the solution is identically zero.

27. (a) 17.08313 (b) 0.48054

Section 7.3 (page 567)

1. (a) $f(t,x) = e^t/x$, $\frac{df}{dt} = \frac{e^t}{x} - \frac{e^{2t}}{x^3}$
(b) $f(t,x) = t(x^2 - x)$, $\frac{df}{dt} = (x^2 - x)(1 + t^2(2x - 1))$
(c) $f(t,x) = te^{x+t} - 1$, $\frac{df}{dt} = t^2e^{2(x+t)} + e^{x+t}$
(d) $f(t,x) = e^{2t} + (1 + \frac{5}{2}e^t)x + x^2$,

$$\frac{df}{dt} = 2e^{2t} + \frac{5}{2}e^t x + (e^{2t} + (1 + \frac{5}{2}e^t)x + x^2)(1 + \frac{5}{2}e^t + 2x)$$

3.

(a)		(b)		(c)		(d)	
t_i	w_i	t_i	w_i	t_i	w_i	t_i	w_i
0.00	1.00000000	0.00	0.50000000	0.00	−1.00000000	0.00	−1.00000000
0.25	1.25000000	0.50	0.46875000	0.67	−1.58491568	0.50	−2.09375000
0.50	1.51252618	1.00	0.37585258	1.33	−1.96969668	1.00	−4.72675623
0.75	1.79455216	1.50	0.23651630	2.00	−1.93770463		
1.00	2.10210368	2.00	0.10527515				

5.

(a)		(b)		(c)		(d)	
t_i	w_i	t_i	w_i	t_i	w_i	t_i	w_i
0.00	0.00000000	π	1.00000000	1.00	1.00000000	0.00	2.00000000
0.50	-0.5416667	$\frac{1}{2}\pi$	−0.31712682	1.25	0.73828125	0.25	2.53401693
1.00	-1.7714318	2π	−0.42291335	1.50	1.04164323	0.50	3.14869947
				1.75	1.93152335	0.75	3.86695803
				2.00	3.61896052	1.00	4.71820994

7. (a)

h	$w_h(1)$	$\frac{\lvert w_{2h}(1)-x(1)\rvert}{\lvert w_h(1)-x(1)\rvert}$
1/1	2.00000000	
1/2	2.08629009	5.309098
1/4	2.10210368	4.754853
1/8	2.10534866	4.357393
1/16	2.10608341	4.170054
1/32	2.10625841	4.082571
1/64	2.10630113	4.040647

(b)

h	$w_h(2)$	$\frac{\lvert w_{2h}(2)-x(2)\rvert}{\lvert w_h(2)-x(2)\rvert}$
1/1	0.00000000	
1/2	0.05273438	1.793373
1/4	0.10527515	4.772374
1/8	0.11611699	4.513314
1/16	0.11847212	4.222684
1/32	0.11902497	4.106664
1/64	0.11915901	4.052527

(c)

h	$w_h(2)$	$\frac{\lvert w_{2h}(2)-x(2)\rvert}{\lvert w_h(2)-x(2)\rvert}$
1/8	−1.74910747	
1/16	−1.69565726	3.013151
1/32	−1.67680542	3.448710
1/64	−1.67117763	3.717553
1/128	−1.66964329	3.859617
1/256	−1.66924324	3.930499
1/512	−1.66914116	3.965494

(d)

h	$w_h(1)$	$\frac{\lvert w_h(1)-w_{h/2}(1)\rvert}{\lvert w_{h/2}(1)-w_{h/4}(1)\rvert}$
1/8	−4.89279075	3.952823
1/16	−4.89724837	3.950537
1/32	−4.89837608	3.969588
1/64	−4.89866153	3.983417
1/128	−4.89873344	3.991367
1/256	−4.89875150	
1/512	−4.89875602	

9. (a)

h	$w_h(1)$	$\frac{\lvert w_h(1)-w_{h/2}(1)\rvert}{\lvert w_{h/2}(1)-w_{h/4}(1)\rvert}$
1/32	-2.13552346	13.692585
1/64	-2.13567324	14.800493
1/128	-2.13568418	15.388943
1/256	-2.13568491	15.691762
1/512	-2.13568496	15.844517
1/1024	-2.13568497	
1/2048	-2.13568497	

(b)

h	$w_h(2\pi)$	$\frac{\lvert w_{2h}(2\pi)-x(2\pi)\rvert}{\lvert w_h(2\pi)-x(2\pi)\rvert}$
1/8	-0.47835633	
1/16	-0.47839156	24.813255
1/32	-0.47839297	20.765249
1/64	-0.47839304	18.507131
1/128	-0.47839304	17.291194
1/256	-0.47839304	16.612934
1/512	-0.47839304	16.946164

(c)

h	$w_h(2)$	$\frac{\lvert w_{2h}(2)-x(2)\rvert}{\lvert w_h(2)-x(2)\rvert}$
1/8	3.61139693	
1/16	3.61112480	20.874128
1/32	3.61111186	18.291339
1/64	3.61111115	17.112113
1/128	3.61111111	16.547795
1/256	3.61111111	16.271921
1/512	3.61111111	16.135227

(d)

h	$w_h(1)$	$\frac{\lvert w_{2h}(1)-x(1)\rvert}{\lvert w_h(1)-x(1)\rvert}$
1/4	4.71820994	
1/8	4.71827684	14.423886
1/16	4.71828150	15.189765
1/32	4.71828181	15.589169
1/64	4.71828183	15.793196
1/128	4.71828183	15.895012
1/256	4.71828183	15.951504

11.

	Euler, $h = 1/16$		Taylor 2nd, $h = 1/8$		Taylor 4th, $h = 1/4$	
t_i	w_i	Error	w_i	Error	w_i	Error
1.00	−1.44269504	0.000000	−1.44269504	0.000000	−1.44269504	0.000000
2.00	−0.33073704	0.029937	−0.37434297	0.013669	−0.37485133	0.014178
3.00	−0.17017295	0.015864	−0.19214629	0.006109	−0.19172169	0.005685
4.00	−0.11021095	0.010014	−0.12373911	0.003515	−0.12338194	0.003157
5.00	−0.07983195	0.007027	−0.08918356	0.002325	−0.08891451	0.002056

13. (a) Rate of convergence is $O(h^4)$
(b) To within roundoff error, the fourth-order Taylor method approximation is exact for any step size
(c) The error term for the fourth-order Taylor method involves the fifth derivative of the solution; in part (b), the fifth derivative of the solution is identically zero.

15. (a) Rate of convergence is $O(h^4)$
(b) To within roundoff error, the fourth-order Taylor method approximation is exact for any step size
(c) The error term for the fourth-order Taylor method involves the fifth deriva-

tive of the solution; in part (b), the fifth derivative of the solution is identically zero.

17. The temperature of the plate is within 1% of the temperature of the furnace at around 51 seconds.

19. At the end of 15 days, roughly 725 boys are confined to bed with the flu. The maximum number of infected boys is roughly 293, occurring around 6.5 days.

Section 7.4 (page 580)

1. For each method

$$w_{i+1} = w_i\left(1 - h + \frac{h^2}{2}\right) + t_i\left(h - \frac{h^2}{2}\right) + \frac{h^2}{2}.$$

Note that $f(t, x)$ is linear in both t and x, so all of the second-order Runge-Kutta methods reduce to the second-order Taylor method.

3. For each method

$$w_{i+1} = w_i(1 - 4h + 8h^2) + t_i\left(\frac{h}{3} - \frac{2h^2}{3}\right) + 2h - \frac{23h^2}{6}.$$

Note that $f(t, x)$ is linear in both t and x, so all of the second-order Runge-Kutta methods reduce to the second-order Taylor method.

5. (a)

t_i	w_i
0.00	1.00000000
0.25	0.79443359
0.50	0.64782017
0.75	0.53202400
1.00	0.43598120

(b)

t_i	w_i
1.00	1.00000000
1.40	1.28260571
1.80	2.81601234
2.20	6.70838939
2.60	14.64752253
3.00	29.10339185

(c)

t_i	w_i
0.00	0.00000000
0.33	−0.47052656
0.67	−2.01761443
1.00	−1.34860048

(d)

t_i	w_i
1.00	0.00000000
1.60	0.55500000
2.20	1.08567603
2.80	1.79226467
3.40	2.97260733
4.00	5.57823118

(e)

t_i	w_i
0.00	0.00000000
0.25	−0.25781250
0.50	−0.55438367
0.75	−0.98016085
1.00	−1.89116857

(f)

t_i	w_i
1.00	2.00000000
1.33	1.58098165
1.67	1.42751269
2.00	1.34784840

7. (a)

t_i	w_i
0.00	1.00000000
0.50	0.65117448
1.00	0.43888130

(b)

t_i	w_i
1.00	1.00000000
2.00	4.96527778
3.00	28.25074074

(c)

t_i	w_i
0.00	0.00000000
0.50	−0.97625289
1.00	0.27524710

(d)

t_i	w_i
1.00	0.00000000
2.50	1.36439657
4.00	5.67411824

(e)

t_i	w_i
0.00	0.00000000
0.50	−0.55116082
1.00	−2.04061543

(f)

t_i	w_i
1.00	2.00000000
1.50	1.49781097
2.00	1.35522770

9. (a)

h	$w_h(5)$	$\frac{\lvert w_{2h}(5)-x(5)\rvert}{\lvert w_h(5)-x(5)\rvert}$
5/16	51.43053935	
5/32	51.40986736	4.842628
5/64	51.40575571	4.242606
5/128	51.40479794	4.087196
5/256	51.40456457	4.036182
5/512	51.40450684	4.016353
5/1024	51.40449248	4.007756

(b)

h	$w_h(2)$	$\frac{\lvert w_{2h}(2)-x(2)\rvert}{\lvert w_h(2)-x(2)\rvert}$
1/16	73.97737456	
1/32	74.52476636	3.636483
1/64	74.67796896	3.815222
1/128	74.71845965	3.907003
1/256	74.72886509	3.953372
1/512	74.73150235	3.976657
1/1024	74.73216618	3.988322

(c)

h	$w_h(0.9)$	$\frac{\lvert w_{2h}(0.9)-x(0.9)\rvert}{\lvert w_h(0.9)-x(0.9)\rvert}$
0.9/16	3.07177583	
0.9/32	3.40358568	2.395580
0.9/64	3.56283111	3.028288
0.9/128	3.61900128	3.514098
0.9/256	3.63543217	3.779628
0.9/512	3.63982794	3.900687
0.9/1024	3.64096008	3.953888

(d)

h	$w_h(2)$	$\frac{\lvert w_h(2)-w_{h/2}(2)\rvert}{\lvert w_{h/2}(2)-w_{h/4}(2)\rvert}$
1/8	-1.05055756	4.695957
1/16	-1.04814837	4.316505
1/32	-1.04763533	4.150401
1/64	-1.04751648	4.073274
1/128	-1.04748784	4.036161
1/256	-1.04748081	
1/512	-1.04747907	

(e)

h	$w_h(2)$	$\frac{\lvert w_h(2)-w_{h/2}(2)\rvert}{\lvert w_{h/2}(2)-w_{h/4}(2)\rvert}$
1/16	3.37040343	3.934035
1/32	3.37021716	3.967252
1/64	3.37016982	3.983689
1/128	3.37015788	3.991860
1/256	3.37015489	3.995934
1/512	3.37015413	
1/1024	3.37015395	

13. $O(h^3)$

15.

	Euler, $h = 1/4$		Heun, $h = 1/2$		RK4, $h = 1$	
t_i	w_i	Error	w_i	Error	w_i	Error
0.00	1.00000000	0.000000	1.00000000	0.000000	1.00000000	0.000000
1.00	1.33902556	0.075188	1.42167812	0.007465	1.41904762	0.004834
2.00	2.17191960	0.064148	2.24114824	0.005080	2.23940304	0.003335
3.00	3.11263856	0.049639	3.16589104	0.003613	3.16464758	0.002370
4.00	4.08339525	0.039710	4.12587976	0.002774	4.12492445	0.001819
5.00	5.06611825	0.032901	5.10126336	0.002244	5.10049047	0.001471

17. (a)

	Taylor 4th, $h = 1/5$		RK4, $h = 1/5$	
t_i	w_i	Error	w_i	Error
1.00	1.00000000	0.000000	1.00000000	0.000000
2.00	3.38633652	0.000042	3.38627234	0.000022
3.00	6.29590332	0.000066	6.29580194	0.000035
4.00	9.54526676	0.000089	9.54513044	0.000047
5.00	13.04730145	0.000112	13.04713066	0.000059
6.00	16.75069118	0.000134	16.75048607	0.000071

(b)

	Taylor 4th, $h = 1/5$		RK4, $h = 1/5$	
t_i	w_i	Error	w_i	Error
0.00	1.00000000	0.000000	1.00000000	0.000000
1.00	1.41306075	0.001153	1.41421742	0.000004
2.00	2.23481119	0.001257	2.23607075	0.000003
3.00	3.16126851	0.001009	3.16227963	0.000002
4.00	4.12229474	0.000811	4.12310714	0.000002
5.00	5.09834937	0.000670	5.09902074	0.000001

(c)

	Taylor 4th, $h = 1/5$		RK4, $h = 1/5$	
t_i	w_i	Error	w_i	Error
1.00	−1.44269504	0.000000	−1.44269504	0.000000
1.80	−0.43983618	0.006125	−0.43373667	0.000025
2.60	−0.23590318	0.002613	−0.23330123	0.000011
3.40	−0.15491306	0.001481	−0.15343868	0.000006
4.20	−0.11284715	0.000972	−0.11187894	0.000004
5.00	−0.08755650	0.000698	−0.08686196	0.000003

19. (a) Rate of convergence is $O(h^4)$
(b) To within roundoff error, the fourth-order Runge-Kutta method approximation is exact for any step size
(c) The error term for the fourth-order Runge-Kutta method involves the fifth derivative of the solution; in part (b), the fifth derivative of the solution is identically zero.

21. (a) Rate of convergence is $O(h^4)$
(b) To within roundoff error, the fourth-order Runge-Kutta method approximation is exact for any step size
(c) The error term for the fourth-order Runge-Kutta method involves the fifth

derivative of the solution; in part (b), the fifth derivative of the solution is identically zero.

23. The new equilibrium level is roughly 1.085; hence, the gene stays on.

25. The temperature of the plate is within 1% of the temperature of the furnace at around 54.4 seconds.

Section 7.5 (page 595)

5. In each case, the optimal RK2 method was used to generate w_1.

(a)

t_i	w_i
0.00	1.00000000
0.25	0.79933449
0.50	0.67246436
0.75	0.56126456
1.00	0.46556988

(b)

t_i	w_i
1.00	1.00000000
1.40	1.28279891
1.80	1.98867507
2.20	5.60037626
2.60	12.33057047
3.00	25.71844588

(c)

t_i	w_i
0.00	0.00000000
0.33	−0.45990485
0.67	−1.31962255
1.00	−6.84444308

(d)

t_i	w_i
1.00	0.00000000
1.60	0.52285714
2.20	0.93913316
2.80	1.47027176
3.40	2.22989974
4.00	3.47208967

(e)

t_i	w_i
0.00	0.00000000
0.25	−0.25520833
0.50	−0.53061930
0.75	−0.88956661
1.00	−1.48298647

(f)

t_i	w_i
1.00	2.00000000
1.33	1.64529267
1.67	1.64788939
2.00	1.53528326

7. (a)

t_i	w_i
0.00	1.00000000
0.17	0.85608541
0.33	0.74386886
0.50	0.65106236
0.67	0.57123580
0.83	0.50108073
1.00	0.43860757

(b)

t_i	w_i
1.00	1.00000000
1.33	0.76567618
1.67	1.55646685
2.00	3.62206377
2.33	7.68756645
2.67	14.98283005
3.00	27.00734411

(c)

t_i	w_i
0.00	0.00000000
0.17	−0.19819551
0.33	−0.48391475
0.50	−0.96996226
0.67	−1.12133256
0.83	−1.55731836
1.00	−33.86018244

(d)

t_i	w_i
1.00	0.00000000
1.50	0.42961951
2.00	0.83123943
2.50	1.30415596
3.00	1.95403769
3.50	3.02101025
4.00	5.25730823

(e)

t_i	w_i
0.00	0.00000000
0.17	−0.16824274
0.33	−0.34686353
0.50	−0.55171559
0.67	−0.81561027
0.83	−1.22576787
1.00	−2.10454269

(f)

t_i	w_i
1.00	2.00000000
1.17	1.74114494
1.33	1.59747693
1.50	1.50570423
1.67	1.44099767
1.83	1.39463924
2.00	1.35962371

9. (a) $O(h^4)$, $\tau_i = -\frac{h^4}{90}y^{(5)}(\xi)$ (b) $O(h^2)$, $\tau_i = \frac{2h^2}{3}y^{(3)}(\xi)$
(c) $O(h^2)$, $\tau_i = \frac{h^2}{3}y^{(3)}(\xi)$ (d) $O(h^4)$, $\tau_i = -\frac{3h^4}{80}y^{(5)}(\xi)$

(e) $O(h^2)$, $\tau_i = -\frac{2h^2}{9}y^{(3)}(\xi)$ (f) $O(h^2)$, $\tau_i = -\frac{5h^2}{8}y^{(3)}(\xi)$

11. (a)

h	$w_h(5)$	$\frac{\lvert w_{2h}(5)-x(5)\rvert}{\lvert w_h(5)-x(5)\rvert}$
5/64	51.40425171	
5/128	51.40446442	10.136597
5/256	51.40448609	14.439962
5/512	51.40448760	15.637848
5/1024	51.40448770	15.929598
5/2048	51.40448770	15.992187
5/4096	51.40448771	16.040270

(b)

h	$w_h(6)$	$\frac{\lvert w_{2h}(6)-x(6)\rvert}{\lvert w_h(6)-x(6)\rvert}$
5/64	16.75062619	
5/128	16.75056238	12.457936
5/256	16.75055721	14.014095
5/512	16.75055684	14.943258
5/1024	16.75055682	15.454346
5/2048	16.75055682	15.725404
5/4096	16.75055682	15.932747

(c)

h	$w_h(5)$	$\frac{\lvert w_{2h}(5)-x(5)\rvert}{\lvert w_h(5)-x(5)\rvert}$
5/64	5.09901364	
5/128	5.09901912	14.865821
5/256	5.09901949	15.763228
5/512	5.09901951	15.967595
5/1024	5.09901951	16.005758
5/2048	5.09901951	16.014934
5/4096	5.09901951	15.965278

(d)

h	$w_h(2)$	$\frac{\lvert w_h(2)-w_{h/2}(2)\rvert}{\lvert w_{h/2}(2)-w_{h/4}(2)\rvert}$
1/8	-1.04752298	9.723000
1/16	-1.04747276	9.762726
1/32	-1.04747792	13.863798
1/64	-1.04747845	15.127961
1/128	-1.04747849	15.608594
1/256	-1.04747849	
1/512	-1.04747849	

(e)

h	$w_h(2)$	$\frac{\lvert w_h(2)-w_{h/2}(2)\rvert}{\lvert w_{h/2}(2)-w_{h/4}(2)\rvert}$
1/16	3.37014881	12.766941
1/32	3.37015349	14.269407
1/64	3.37015386	15.100320
1/128	3.37015388	15.540741
1/256	3.37015388	15.780188
1/512	3.37015388	
1/1024	3.37015388	

13. (a)

	AB2, $h = 0.01$		Taylor 2nd, $h = 0.02$	
t_i	w_i	Error	w_i	Error
0.00	0.00000000	0.000000	0.00000000	0.000000
0.30	0.27025337	0.000048	0.27022260	0.000079
0.60	0.59029138	0.000382	0.59006321	0.000610
0.90	3.41762626	0.223717	3.33203352	0.309310

(b)

	AB2, $h = 1/8$		Taylor 2nd, $h = 1/4$	
t_i	w_i	Error	w_i	Error
1.00	−1.44269504	0.000000	−1.44269504	0.000000
2.00	−0.38570933	0.025036	−0.44610921	0.085435
3.00	−0.19773794	0.011701	−0.22285703	0.036820
4.00	−0.12702571	0.006801	−0.14090817	0.020684
5.00	−0.09137321	0.004514	−0.10033612	0.013477

(c)

	AB2, $h = 1/8$		Taylor 2nd, $h = 1/4$	
t_i	w_i	error	w_i	error
0.00	−0.50000000	0.000000	−0.50000000	0.000000
1.00	−1.26848940	0.000452	−1.26807529	0.000866
2.00	−2.11900755	0.000195	−2.11882837	0.000375
3.00	−3.04751103	0.000085	−3.04756672	0.000141

(d)

	AB2, $h = 1/16$		Taylor 2nd, $h = 1/8$	
t_i	w_i	Error	w_i	Error
1.00	0.00000000	0.000000	0.00000000	0.000000
1.25	1.79613452	0.088780	1.73559475	0.149320
1.50	8.51652599	0.410723	8.30280381	0.624445
1.75	27.29017539	1.187193	26.71671192	1.760657
2.00	71.94880689	2.783581	70.63811735	4.094271

15. (a)

	AB4, $h = 1/40$		RK4, $h = 1/10$	
t_i	w_i	Error	w_i	Error
0.00	0.00000000	0.000000	0.00000000	0.000000
0.30	0.27029871	0.000003	0.27030138	0.000000
0.60	0.59060273	0.000070	0.59067214	0.000001
0.90	3.36736561	0.273978	3.49022244	0.151121

(b)

	AB4, $h = 1/8$		RK4, $h = 1/2$	
t_i	w_i	Error	w_i	Error
1.00	−1.44269504	0.000000	−1.44269504	0.000000
2.00	−0.36874827	0.008075	−0.34917427	0.011499
3.00	−0.18919231	0.003155	−0.18141971	0.004617
4.00	−0.12198821	0.001764	−0.11764541	0.002579
5.00	−0.08801030	0.001151	−0.08517270	0.001686

(c)

	AB4, $h = 1/16$		RK4, $h = 1/4$	
t_i	w_i	Error	w_i	Error
0.00	−0.50000000	0.00000000	−0.50000000	0.00000000
1.00	−1.26894188	0.00000046	−1.26894214	0.00000072
2.00	−2.11920278	0.00000014	−2.11920472	0.00000180
3.00	−3.04742567	0.00000021	−3.04742782	0.00000195

(d)

	AB4, $h = 1/32$		RK4, $h = 1/8$	
t_i	w_i	Error	w_i	Error
1.00	0.00000000	0.000000	0.00000000	0.000000
1.25	1.88479397	0.000121	1.88383671	0.001078
1.50	8.92668601	0.000563	8.92331721	0.003932
1.75	28.47584661	0.001522	28.46770490	0.009664
2.00	74.72906016	0.003328	74.71271684	0.019671

17. (a)

	Adams PC4, $h = 1/40$		RK4, $h = 1/20$	
t_i	w_i	Error	w_i	Error
0.00	0.00000000	0.000000	0.00000000	0.00000000
0.30	0.27029871	0.000003	0.27030124	0.00000001
0.60	0.59060273	0.000070	0.59067321	0.00000015
0.90	3.36736561	0.273978	3.61653462	0.02480874

(b)

	Adams PC4, $h = 1/4$		RK4, $h = 1/2$	
t_i	w_i	Error	w_i	Error
1.00	−1.44269504	0.000000	−1.44269504	0.000000
2.00	−0.42998497	0.069311	−0.34917427	0.011499
3.00	−0.23542897	0.049392	−0.18141971	0.004617
4.00	−0.14216907	0.021944	−0.11764541	0.002579
5.00	−0.09895853	0.012100	−0.08517270	0.001686

(c)

	Adams PC4, $h = 1/8$		RK4, $h = 1/4$	
t_i	w_i	Error	w_i	Error
0.00	−0.50000000	0.000000	−0.50000000	0.000000
1.00	−1.26894754	0.000006	−1.26894214	0.000001
2.00	−2.11920048	0.000002	−2.11920472	0.000002
3.00	−3.04742198	0.000004	−3.04742782	0.000002

(d)

	Adams PC4, $h = 1/16$		RK4, $h = 1/8$	
t_i	w_i	Error	w_i	Error
1.00	0.00000000	0.000000	0.00000000	0.000000
1.25	1.88426289	0.000652	1.88383671	0.001078
1.50	8.92125957	0.005990	8.92331721	0.003932
1.75	28.45927622	0.018092	28.46770490	0.009664
2.00	74.69080213	0.041586	74.71271684	0.019671

19. (a) Rate of convergence is $O(h^4)$
(b) To within roundoff error, the approximate solution is exact for any step

size
(c) The error term for the four-step Adams-Bashforth method involves the fifth derivative of the solution; in part (b), the fifth derivative of the solution is identically zero.

21. (a) Rate of convergence is $O(h^4)$
(b) To within roundoff error, the approximate solution is exact for any step size
(c) The error term for the four-step Adams-Bashforth method involves the fifth derivative of the solution; in part (b), the fifth derivative of the solution is identically zero.
(d) Results are identical to parts (a), (b) and (c).

23. Using the four-step Adams-Bashforth method:

z	0.000	0.005	0.010	0.015	0.020	0.025	0.030	0.035	0.040
M_c	0.0160	0.0165	0.0171	0.0176	0.0182	0.0188	0.0194	0.0200	0.0207
z	0.045	0.050	0.055	0.060	0.065	0.070	0.075	0.080	0.085
M_c	0.0213	0.0220	0.0228	0.0235	0.0243	0.0250	0.0258	0.0267	0.0275
z	0.090	0.095	0.100	0.105	0.110	0.115	0.120	0.125	0.130
M_c	0.0284	0.0293	0.0303	0.0313	0.0323	0.0333	0.0344	0.0355	0.0366
z	0.135	0.140	0.145	0.150	0.155	0.160	0.165	0.170	0.175
M_c	0.0378	0.0390	0.0402	0.0415	0.0429	0.0442	0.0456	0.0471	0.0486
z	0.180	0.185	0.190	0.195	0.200				
M_c	0.0501	0.0517	0.0534	0.0550	0.0568				

Section 7.6 (page 607)

3. Since $\phi(f, t_i, w_i, h) = \frac{1}{4}f(t_i, w_i) + \frac{3}{4}f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3}f(t_i, w_i)\right)$ it follows that $\phi(f, t_i, y_i, 0) = f(t_i, w_i)$ and the optimal RK2 method is consistent. Furthermore, ϕ is Lipschitz with constant $L_\phi = L + \frac{h_0 L^2}{2}$; consequently, the method is stable and convergent. Finally, since the local truncation error is $O(h^2)$, it follows that the global error is also $O(h^2)$.

5. Since $\phi(f, t_i, w_i, h) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \cdots + \frac{h^{p-1}}{p!}f^{(p)}(t_i, w_i)$, it follows that $\phi(f, t_i, y_i, 0) = f(t_i, w_i)$ and the Taylor method of order p is consistent. Furthermore, ϕ is Lipschitz with constant $L_\phi = L\left(1 + \frac{h}{2} + \cdots + \frac{h^{p-1}}{p!}\right)$; consequently, the method is stable and convergent. Finally, since the local truncation error is $O(h^p)$, it follows that the global error is also $O(h^p)$.

7. Since $a_1 + a_2 = 1$ and $-a_1 - 2a_2 + b_0 = 0$, the method is consistent. The characteristic polynomial for the method is $\lambda^2 - \frac{4}{3}\lambda + \frac{1}{3}$, whose roots are 1 and $\frac{1}{3}$. Hence, the method is stable (strongly) and convergent.

9. Since $a_1 + a_2 = 1$ and $-a_1 - 2a_2 + b_0 + b_1 + b_2 = 0$, the method is consistent. The characteristic polynomial for the method is $\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2}$, whose roots are 1 and $-\frac{1}{2}$. Hence, the method is stable (strongly) and convergent.

11. Since $a_1 + a_2 = 1$ and $-a_1 - 2a_2 + b_2 = 0$, the method is consistent. The characteristic polynomial for the method is $\lambda^2 - 4\lambda + 3$, whose roots are 1 and 3. Hence, the method is neither stable nor convergent.

15. "Sawtooth" oscillations begin around $t = 15$.

Section 7.7 (page 621)

1. (a)

t_i	w_i
1.112963	1.232078
1.217664	1.457464
1.332912	1.715946

(b)

t_i	w_i
0.250000	1.030776
0.417314	1.083582
0.583334	1.157704

(c)

t_i	w_i
0.131556	0.302483
0.260969	0.359698
0.408572	0.430094

(d)

t_i	w_i
0.031687	−1.456815
0.059247	−1.423842
0.090068	−1.391101

(e)

t_i	w_i
0.090799	0.090056
0.189868	0.183261
0.298554	0.274120

3. (a)

t_i	w_i
1.045957	1.092955
1.091915	1.187930
1.137872	1.284840
1.183830	1.383606
1.229787	1.484157
1.275744	1.586426

(b)

t_i	w_i
0.033408	1.000558
0.066817	1.002230
0.100225	1.005010
0.133634	1.008889
0.167042	1.013856
0.200451	1.019892

(c)

t_i	w_i
0.038669	0.264779
0.077339	0.280106
0.116008	0.295963
0.154678	0.312328
0.193347	0.329175
0.229660	0.345406

(d)

t_i	w_i
0.006714	−1.490318
0.013428	−1.480934
0.020143	−1.471835
0.026857	−1.463006
0.033571	−1.454436
0.040285	−1.446112

(e)

t_i	w_i
0.017138	0.017133
0.034275	0.034235
0.051413	0.051278
0.068551	0.068230
0.085688	0.085064
0.102826	0.101750

5. (a)

t_i	w_i
1.045367	1.091753
1.085434	1.174418
1.127126	1.262009

(b)

t_i	w_i
0.096570	1.004653
0.186522	1.017248
0.286991	1.040370

(c)

t_i	w_i
0.079362	0.280922
0.146974	0.309028
0.221516	0.341732

(d)

t_i	w_i
0.006999	−1.489913
0.013829	−1.480383
0.020904	−1.470820

(e)

t_i	w_i
0.069104	0.068775
0.113738	0.112285
0.166335	0.161857

7. (a)

t_i	w_i
1.065145	1.132324
1.130290	1.268659
1.195435	1.408752
1.260580	1.552397

(b)

t_i	w_i
0.084390	1.003561
0.168781	1.014131
0.253171	1.031508
0.337562	1.055359

(c)

t_i	w_i
0.080373	0.281302
0.160746	0.314894
0.241119	0.350534
0.321493	0.387918

(d)

t_i	w_i
0.008082	−1.488386
0.016164	−1.477194
0.024245	−1.466407
0.032327	−1.456001

(e)

t_i	w_i
0.024593	0.024578
0.049185	0.049059
0.073778	0.073363
0.098370	0.097406

9. Although $x(t) \approx 1$ for $t > 0.05$, the time steps remain between roughly 0.03 and 0.04.

11. Although $x(t)$ is roughly linear for $t > 0.05$, the time steps remain between roughly 0.10 and 0.15.

13. RKF45: 66 function evaluations; RKV56: 56 function evaluations; VS_PC4: 72 function evaluations.

15. RKF45: 954 function evaluations; RKV56: 1144 function evaluations; VS_PC4: 930 function evaluations.

17. RKF45: 84 function evaluations; RKV56: 72 function evaluations; VS_PC4: 104 function evaluations.

Section 7.8 (page 631)

1. (a)

t_i	x_i	y_i
0.25	1.750000	−1.000000
0.50	1.500000	−0.937500
0.75	1.265625	−0.843750
1.00	1.054688	−0.738281

(b)

t_i	x_i	y_i
0.25	0.750000	0.250000
0.50	0.546875	0.433594
0.75	0.356155	0.549933
1.00	0.155121	0.597393

(c)

t_i	x_i	y_i	z_i
0.25	1.750000	−1.250000	0.750000
0.50	2.812500	−1.687500	0.562500
0.75	4.359375	−2.390625	0.421875
1.00	6.644531	−3.480469	0.316406

(d)

t_i	x_i	y_i	z_i
0.25	1.000000	0.625000	1.000000
0.50	0.906250	0.343750	0.906250
0.75	0.765625	0.165771	0.757568
1.00	0.615662	0.075028	0.599906

3. (a)

t_i	x_i	y_i
0.25	1.752279	−0.973470
0.50	1.516294	−0.909751
0.75	1.298973	−0.826592
1.00	1.103606	−0.735712

(b)

t_i	x_i	y_i
0.25	0.774353	0.220174
0.50	0.575159	0.380959
0.75	0.376635	0.479216
1.00	0.160449	0.515125

(c)

t_i	x_i	y_i	z_i
0.25	1.938314	−1.358561	0.778809
0.50	3.420947	−2.013745	0.606543
0.75	5.815040	−3.143710	0.472381
1.00	9.722662	−5.045278	0.367894

(d)

t_i	x_i	y_i	z_i
0.25	0.960566	0.673079	0.958149
0.50	0.868592	0.441209	0.856921
0.75	0.755415	0.290916	0.731669
1.00	0.641364	0.200770	0.606911

5. (a)

t_i	x_i	y_i
0.25	1.752279	-0.973470
0.50	1.516294	-0.909751
0.75	1.298973	-0.826592
1.00	1.103081	-0.734979
1.25	0.930638	-0.643869
1.50	0.780168	-0.556664

(b)

t_i	x_i	y_i
0.25	0.774353	0.220174
0.50	0.575159	0.380959
0.75	0.376635	0.479216
1.00	0.161088	0.513028
1.25	-0.084025	0.492892
1.50	-0.359521	0.414141

(c)

t_i	x_i	y_i	z_i
0.25	1.938314	-1.358561	0.778809
0.50	3.420947	-2.013745	0.606543
0.75	5.815040	-3.143710	0.472381
1.00	9.684089	-5.026095	0.368101
1.25	15.987666	-8.137218	0.286769
1.50	26.307773	-13.265638	0.223503

(d)

t_i	x_i	y_i	z_i
0.25	0.960566	0.673079	0.958149
0.50	0.868592	0.441209	0.856921
0.75	0.755415	0.290916	0.731669
1.00	0.641156	0.197267	0.609411
1.25	0.534549	0.145231	0.493986
1.50	0.445351	0.119071	0.399363

7. (a) Let $u_1 = x$, $u_2 = x'$, and $u_3 = x''$. Then $u_1' = u_2$, $u_2' = u_3$, and $u_3' = -4u_3 - 5u_2$.
(b) Let $u_1 = x$, $u_2 = x'$, $u_3 = x''$, and $u_4 = x'''$. Then $u_1' = u_2$, $u_2' = u_3$, $u_3' = u_4$, and $u_4' = u_1$.
(c) Let $u_1 = x$, $u_2 = x'$, and $u_3 = x''$. Then $u_1' = u_2$, $u_2' = u_3$, and

$u_3' = -\frac{1}{2}u_1u_3$.
(d) Let $u_1 = x$, $u_2 = x'$, and $u_3 = x''$. Then $u_1' = u_2$, $u_2' = u_3$, and $u_3' = u_2\ln(u_3) + \sin(u_1)$.
(e) Let $u_1 = x$ and $u_2 = x'$. Then $u_1' = u_2$ and $u_2' = 1 - \sin(u_1u_2)$.
(f) Let $u_1 = x$, $u_2 = x'$, $u_3 = x''$, and $u_4 = x'''$. Then $u_1' = u_2$, $u_2' = u_3$, $u_3' = u_4$, and $u_4' = \cos(u_4) - \sqrt{u_1u_3}$.

9. (a)

t_i	x_i	x_i'	x_i''
0.25	0.000000	0.250000	1.000000
0.50	0.062500	0.500000	1.000000
0.75	0.187500	0.750000	0.992188
1.00	0.375000	0.998047	0.968933

(b)

t_i	x_i	x_i'	x_i''
0.25	1.000000	−0.250000	0.000000
0.50	0.937500	−0.250000	0.312500
0.75	0.875000	−0.171875	0.312500
1.00	0.832031	−0.093750	0.214844

(c)

t_i	x_i	x_i'
0.25	0.525000	−0.006250
0.50	0.523438	−0.138632
0.75	0.488780	−0.294653
1.00	0.415116	−0.472913

(d)

t_i	x_i	x_i'
0.25	1.000000	−0.250000
0.50	0.937500	−0.188149
0.75	0.890463	−0.114519
1.00	0.861833	−0.052206

11. (a)

t_i	x_i	x_i'	x_i''
0.25	0.031250	0.249919	0.998699
0.50	0.124878	0.498702	0.989644
0.75	0.280280	0.743479	0.965516
1.00	0.495915	0.979697	0.920363

(b)

t_i	x_i	x_i'	x_i''
0.25	0.977376	−0.149740	−0.2879232
0.50	0.934890	−0.175915	0.029210
0.75	0.893290	−0.151642	0.140117
1.00	0.860030	−0.113507	0.153877

(c)

t_i	x_i	x_i'
0.25	0.510637	−0.019033
0.50	0.488550	−0.161247
0.75	0.428281	−0.324286
1.00	0.324666	−0.508194

(d)

t_i	x_i	x_i'
0.25	0.978673	−0.131815
0.50	0.943086	−0.138463
0.75	0.912491	−0.101970
1.00	0.892660	−0.056670

13. (a)

t_i	x_i	x_i'	x_i''
0.25	0.031250	0.249919	0.998699
0.50	0.124878	0.498702	0.989644
0.75	0.280280	0.743479	0.965516
1.00	0.496060	0.979586	0.919958
1.25	0.769130	1.201053	0.850036
1.50	1.094992	1.402008	0.756631

(b)

t_i	x_i	x_i'	x_i''
0.25	0.977376	-0.149740	-0.2879232
0.50	0.934890	-0.175915	0.029210
0.75	0.893290	-0.151642	0.140117
1.00	0.856814	-0.106539	0.142084
1.25	0.835210	-0.072999	0.115945
1.50	0.816912	-0.042630	0.085962

(c)

t_i	x_i	x_i'
0.25	0.510637	-0.019033
0.50	0.488550	-0.161247
0.75	0.428281	-0.324286
1.00	0.324882	-0.508051
1.25	0.172748	-0.714990
1.50	-0.034510	-0.942232

(d)

t_i	x_i	x_i'
0.25	0.978673	-0.131815
0.50	0.943086	-0.138463
0.75	0.912491	-0.101970
1.00	0.888364	-0.040857
1.25	0.886617	-0.029003
1.50	0.868791	0.061348

15. Using the RKF45 method (with $h_{\min} = 0.001$, $h_{\max} = 0.5$ and $TOL = 5 \times 10^{-7}$), $F(30) = 0.057463$ and $C(30) = 0.347206$.

17.

θ_0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
period	2.009	2.012	2.019	2.027	2.039	2.053	2.070	2.091	2.114	2.141
θ_0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
period	2.170	2.204	2.243	2.285	2.332	2.385	2.445	2.510	2.584	2.668

19.

ϵ	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00
period	7.630	7.920	8.225	8.540	8.856	9.190	9.520	9.860	10.210
ϵ	4.25	4.50	4.75	5.00	5.25	5.50	5.75	6.00	
period	10.550	10.900	11.260	11.610	11.974	12.330	12.698	13.062	

21. Using the RKF45 method (with $h_{\min} = 0.001$, $h_{\max} = 0.5$ and $TOL = 5 \times 10^{-7}$), $x(20) = 3.563594$, $x'(20) = 4.709370 \times 10^{-7}$ and $x''(20) = -4.432607 \times 10^{-7}$.

23. Using the RKF45 method (with $h_{\min} = 0.001$, $h_{\max} = 0.5$ and $TOL = 5 \times 10^{-7}$), $\theta(30) = 0.250327$, $\theta'(30) = 0.018103$, $x(30) = 2.336166$, and $z(30) = 29.826942$.

Section 7.9 (page 652)

1. (a) $Q(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$ (b) $-3.2170 < h\lambda < 0$
3. (a) $Q(z, h\lambda) = z^4 - (1 + \frac{55}{24}h\lambda)z^3 + \frac{59}{24}h\lambda z^2 - \frac{37}{24}h\lambda z + \frac{9}{24}h\lambda$
5. (a) $Q(z, h\lambda) = z^4 - \frac{8}{3}h\lambda z^3 + \frac{4}{3}h\lambda z^2 - \frac{8}{3}h\lambda z - 1$
7. (a) $Q(z, h\lambda) = (1 - \frac{2}{3}h\lambda)z^2 - \frac{4}{3}z + \frac{1}{3}$
   (b) The roots of $Q(z, h\lambda)$ are $(2 \pm \sqrt{1 + 2h\lambda})/(3 - 2h\lambda)$.
   (c) When $h\lambda < -\frac{1}{2}$, $1 + 2h\lambda < 0$.
9. In each case, Newton's method was terminated with a convergence tolerance of 5×10^{-7}.

(a)

t_i	w_i
0.00	1.00000000
0.25	0.79155430
0.50	0.64428813
0.75	0.52825556
1.00	0.43211485

(b)

t_i	w_i
1.00	1.00000000
1.40	0.74347636
1.80	2.20601958
2.20	5.87415138
2.60	13.43032779
3.00	27.34534677

(c)

t_i	w_i
0.00	0.00000000
0.33	−0.53634398
0.67	−1.56173287
1.00	−56.18855622

(d)

t_i	w_i
1.00	0.00000000
1.60	0.54272902
2.20	1.08124238
2.80	1.85143514
3.40	3.48686151
4.00	6.44425098

(e)

t_i	w_i
0.00	0.00000000
0.25	−0.25895147
0.50	−0.56703514
0.75	−1.09635152
1.00	−1.14337706

(f)

t_i	w_i
1.00	2.00000000
1.33	1.53334437
1.67	1.38613456
2.00	1.31209391

11. (a) $Q(h\lambda) = 1 + h\lambda$
    (b) For IVP#1, with $t_f = 0.5$ and $h = \frac{1}{700} > \frac{1}{1000}$, $Q(h\lambda) = -\frac{3}{7}$ and spurious oscillations are apparent for $0 < t < 0.02$. For IVP#2, with $t_f = 1$ and $h = \frac{1}{25} > \frac{1}{39}$, $Q(h\lambda) = -\frac{14}{25}$ and spurious oscillations are apparent for $0 < t < 0.3$.
13. (a) For the first initial value problem, absolute stability requires $h < \frac{1}{200}$; for the second and third problems, absolute stability requires $h < \frac{1}{39}$.

15. The backward Euler method introduces no spurious oscillations into the approximate solution. Both the amplitude and the duration of the oscillations introduced by the Trpaezoidal method increase with increasing h. For the second-order backward differentiation formula, the amplitude and duration of the oscillations decrease with increasing h.