Logic, Sets

Section 1.1 Logic

Introduction

- Applications of discrete mathematics:
  - Formal Languages (computer languages)
  - Digital Circuits
  - Compiler Design
  - Data Structures
  - Computability
  - Algorithm Design
  - Relational Database Theory
  - Complexity Theory (counting)

Example (counting)

- The Traveling Salesman Problem, important in
  - Circuit design
  - Many other CS problems
- Given:
  - n cities $c_1, c_2, \ldots, c_n$
  - Distance between city i and j, d_{ij}
  - Find the shortest tour.

Example (cont)

- Assume a very fast PC:
  - 1 flop = 1 nanosecond = 10^{-9} sec.
    = 1,000,000,000 ops/sec = 1 GHz.
- A tour requires n-1 additions. How many different tours?
  - Choose the first city n ways,

  the second city n-1 ways,

  the third city n-2 ways,

  etc.

Example (cont)

- \# tours = n (n-1) (n-2) . . . .(2) (1) = n! (*Combinations*) Total number of additions = (n-1) n! (*Rule of Product*)
- If n=8, T(n) = 7•8! = 282,240 flops < 1/3 second.
- HOWEVER, If n=50, T(n) = 49•50! = 1.48 10^{66}

  = 1.49 10^{57} seconds

  = 2.48 10^{55} minutes

  = 4.13 10^{53} hours

  = 1.72 10^{52} days

  = 2.46 10^{51} weeks

  = 4.73 10^{49} years.

  ...a long time.

Logic: Definitions

- **Proposition**: a proposition is a statement that is either true (T or 1) or false (F or 0), but not both.

  1 + 1 = 2.

  2 + 2 = 5.

  ‘the moon is made of green cheese’

  ‘ go to town!’

  ‘What time is it?’

  - A fact-based declaration is a proposition, even if no one knows whether it is true.

  11213 is prime.

  1 is prime.

  There exists an odd perfect number.

Logic: Definitions

- **Boolean Variable**: a variable whose value is either true (1) or false (0).
- **Propositional variables**: P, Q, R, S, . . .
- New Propositions from old: **calculus of propositions** (**propositional logic**) - relate new propositions (**compound proposition**) to old ones.
- Logical analysis begins with the modeling of fact-based natural language declarations by propositions.
- Examples:
  - Portland is the capital of Oregon.
  - Columbia University was founded in 1754.
  - If 2+2 = 5, then you are the pope.

Logic: Definitions

- **Truth Table:** used to display the relationship between the truth values of propositions.
  - The boolean domain is the set {T or 1, F or 0}.
  - Either of its elements is called a boolean value.

Logical Operations - Negation

- ***Negation***

  ‘not’

  Negation of P: it is not the case that P

  Symbol: $\neg$

  Truth Table:

- Example:

  P: I am going to town.

  $\neg$ P:

  It is not the case that I am going to town;

  I am not going to town;

Logical Operations - Conjunction

- ***Conjunction***
  - ‘and’
  - “P and Q” is true when both P and Q are true, and is false otherwise.
  - Symbol: $\wedge$
  - Truth Table:
- Example:
  - P - ‘I am going to town’
  - Q - ‘It is going to rain’
  - $P \wedge Q$: ‘I am going to town and it is going to rain.’
  - Note: Both P and Q must be true!!!!!

Logical Operations - Disjunction

- ***Disjunction***

  inclusive ‘or’

  “P or Q” is false when P and Q are both false, and true otherwise.

  Symbol: $\vee$

  Truth Table:

- Example:

  P - ‘I am going to town’

  Q - ‘It is going to rain’

  $P \vee Q$: ‘I am going to town or it is going to rain.’

  Note: Only one of P, Q need be true. Hence, the *inclusive* nature.

Logical Operations-Exclusive OR

- ***Exclusive OR***

  Symbol:⊕

  Exclusive OR of P and Q, is the proposition that is true when exactly one of P and Q is true, and is false otherwise.

  Truth Table:

- Example:

  P - ‘I am going to town’

  Q - ‘It is going to rain’

  P ⊕ Q: ‘Either I am going to town or it is going to rain.’

  Truth Table:

  Note: Only one of P and Q must be true.

Logical Operations - Others

- ***Bitwise AND, OR operations***
- ***Others: NAND (|); NOR (↓)***

Logical Operations - Implication

- ***Implication***
  - ‘If...then...’
  - Symbol: $\rightarrow$
  - Implication P $\rightarrow$ Q is the proposition that is false when P is true and Q is false; and true otherwise.
  - Truth Table:

P	Q	P $\rightarrow$ Q
0	0	1
0	1	1
1	0	0
1	1	1

Implication

- *Terminology*:
  P = premise, hypothesis
  Q = conclusion, consequence
- Example:
  P - 'I am going to town'
  Q - 'It is going to rain'
  $P \rightarrow Q$: 'If I am going to town then it is going to rain.'

Implication

- *Equivalent forms:*
  - If P, then Q
  - P implies Q
  - If P, Q
  - P only if Q
  - P is a sufficient condition for Q
  - Q if P
  - Q whenever P
  - Q is a necessary condition for P

Implication

- Note: The implication is false only when P is true and Q is false!

  ‘If the moon is made of green cheese then I have more money than Bill Gates’

  ‘If the moon is made of green cheese then I’m on welfare’

  ‘If 1+1=3 then your grandma wears combat boots’

  ‘If I’m wealthy then the moon is not made of green cheese.’

  ‘If I’m not wealthy then the moon is not made of green cheese.’

Implication

- Converse and Contrapositive:

  $Q \rightarrow P$ is the ***CONVERSE*** of $P \rightarrow Q$

  $\neg Q \rightarrow \neg P$ is the ***CONTRAPOSITIVE*** of $P \rightarrow Q$

Implication

- Example: Find the converse and contrapositive of the following statement:
  R: ‘Raining tomorrow is a sufficient condition for my not going to town.’
  Step 1: Assign propositional variables to component propositions
  P: It will rain tomorrow
  Q: I will not go to town
  Step 2: Symbolize the assertion
  R: $P \rightarrow Q$
  Step 3: Symbolize the converse
  $Q \rightarrow P$
  Step 4: Convert the symbols back into words
  ‘If I don’t go to town then it will rain tomorrow’
  Or
  ‘Raining tomorrow is a *necessary condition* for my not going to town.’
  or
  ‘My not going to town is a *sufficient condition* for it raining tomorrow.’

Logical Operations - Bidirectional

- ***Biconditional (Bidirectional)***

  'if and only if', 'iff'

  Symbol: $\leftrightarrow$

  Biconditinal P$\leftrightarrow$ Q is the proposition that is true only when P and Q have the same truth value.

  Truth Table:

- Example:

  P - 'I am going to town', Q - 'It is going to rain'

  P$\leftrightarrow$Q: 'I am going to town if and only if it is going to rain.'

  Note: Both P and Q must have the same truth value.

  Equivalent forms:

  P if and only if Q

  P is necessary and sufficient for Q

Form a Logical Expression

- Translating a sentence into a propositional expression:
  - Step 1: breaking assertions into component propositions
  - Step 2: look for the logical operators.
- Example:

  'If I go to Harry's or go to the country I will not go shopping.'

  P: I go to Harry's

  Q: I go to the country

  R: I will go shopping

Form a Logical Expression

- Example:

  ‘You can not drink beer if you are under 21 unless you are accompanied by your teacher’

  - Step 1:
  - Step 2:

Truth Table

- Constructing a truth table:
  - one column for each propositional variable
  - one for the compound proposition
  - count in binary

  You may find it easier to include columns for propositions which themselves are compound propositions.
- Example: $(P \vee Q) \rightarrow \neg R$