Logic, Sets

Section 1.2 Propositional Equivalences

Definitions

- A *tautology* is a proposition which is always true.
  Classic Example: $P \vee \neg P$
- A *contradiction* is a proposition which is always false.
  Classic Example: : $P \wedge \neg P$
- A *contingency* is a proposition which neither a tautology nor a contradiction.
  Example: $(P \vee Q) \rightarrow \neg R$

Logical Equivalence: Definition

- Two propositions P and Q are *logically equivalent* if $P \leftrightarrow Q$ is a tautology.
  We write: $P \Leftrightarrow Q$, or, $P \equiv Q$
- Example:
  $(P \rightarrow Q) \wedge (Q \rightarrow P) \Leftrightarrow (P \leftrightarrow Q)$

  Note: Because of this equivalence, *if and only if* or *iff* is also stated as *is a necessary and sufficient condition for.*

Logical Equivalence: Proof

- To show logical equivalence, the left side and the right side must have the same truth values independent of the truth value of the component propositions.
- Example:
  Show that $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

Logical Equivalence: Proof

- To show a proposition is not a tautology or two propositions are not logical equivalent: use an *abbreviated* truth table.
  - Try to find a *counter example* or to *disprove* the assertion.
  - Search for a case where the proposition is false.
- Example: find out either the two expression are logical equivalent: $P \rightarrow Q$ *and* $P \vee \neg Q$

Logical Equivalence: Proof

- Example:
  Prove $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

Logical Equivalence Table

Some famous logical equivalences

$P \wedge T \Leftrightarrow P$ $P \vee F \Leftrightarrow P$	Identity
$P \vee T \Leftrightarrow T$ $P \wedge F \Leftrightarrow F$	Domination
$P \vee P \Leftrightarrow P$ $P \wedge P \Leftrightarrow P$	Idempotency
$\neg(\neg P) \Leftrightarrow P$	Double negation
$P \vee Q \Leftrightarrow Q \vee P$ $P \wedge Q \Leftrightarrow Q \wedge P$	Commutativity

Logical Equivalence Table

$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	Associativity
$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	Distributivity
$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	DeMorgan's laws
$P \rightarrow Q \Leftrightarrow \neg P \vee Q$	Implication
$P \vee \neg P \Leftrightarrow T$	Tautology

Logical Equivalence Table

$P \wedge \neg P \Leftrightarrow F$	Contradiction
$(P \rightarrow Q) \wedge (Q \rightarrow P) \Leftrightarrow (P \leftrightarrow Q)$	Equivalence
$(P \rightarrow Q) \wedge (P \rightarrow \neg Q) \Leftrightarrow \neg P$	Absurdity
$(P \rightarrow Q) \Leftrightarrow (\neg Q \rightarrow \neg P)$	Contrapositive
$P \vee (P \wedge Q) \Leftrightarrow P$ $P \wedge (P \vee Q) \Leftrightarrow P$	Absorption
$(P \wedge Q) \rightarrow R \Leftrightarrow P \rightarrow (Q \rightarrow R)$	Exportation

Logical Equivalence

- Equivalent expressions can always be substituted for each other in a more complex expression - useful for simplification and logic proof.
- Example:
  Prove using known logical equivalences:
  $(P \rightarrow Q) \Leftrightarrow (\neg Q \rightarrow \neg P)$