Logic, Sets

Section 1.3 Predicates and Quantifiers

Predicate

- A generalization of propositions - ***propositional functions*** or ***predicates***: propositions which contain variables.
- Examples:
  - Let U = Z, the integers = {. . . -2, -1, 0 , 1, 2, 3, ...}

    P(x): x > 0 is the predicate.
    It has no truth value until the variable x is bound.
  - P(x, y): x + y = 3
  - Q(x): x studies computer science

Predicate

- Predicates become propositions once every variable is *bound* by
  - assigning it a value from the *Universe of Discourse* U, or
  - quantifying it.
- Example: Let U = Z, the integers = {. . . -2, -1, 0 , 1, 2, 3, . . .}, P(x): x > 0 is the predicate
  - P(-3) is false,
  - P(0) is false,
  - P(3) is true.

Predicate

- Example: Let U = Z, the integers = {. . . -2, -1, 0 , 1, 2, 3, . . .}, P(x): x > 0 is the predicate
  - $P(y) \vee \neg P(0)$
  - $P(3) \vee \neg P(0)$
- Example:

  Let R be the three-variable predicate R(x, y z): x + y= z

  Find the truth value of

  R(2, -1, 5), R(3, 4, 7), R(x, 3, z)

Quantifiers

- Universal: P(x) is true for every x in the universe of discourse.
  - Notation: *universal quantifier*

  $\forall x P(x)$

  'For all x, P(x)', 'For every x, P(x)'
  - The variable x is bound by the universal quantifier producing a proposition.
- Example: U={1,2,3}

  $$\forall x P(x) \Leftrightarrow P(1) \wedge P(2) \wedge P(3)$$

Quantifiers

- Existential: P(x) is true for some x in the universe of discourse.
  - Notation: *existential quantifier* $\exists x P(x)$

  'There is an x such that P(x),' 'For some x, P(x)', 'For at least one x, P(x)', 'I can find an x such that P(x).'
- *Example 1: U={1,2,3}*

  $\exists x P(x) \Leftrightarrow P(1) \vee P(2) \vee P(3)$
- *Example 2: P(x): x > 3 (x $\in$ R)*

  $\forall x P(x)$

  $\exists x P(x)$

Quantifiers

- A predicate is not a proposition until *all* variables have been bound either by quantification or assignment of a value!
- Example: Let U = R, the real numbers, P(x,y): xy= 0
  Find the truth value of each case.

  $$\forall x \forall y P(x, y)$$
  $$\forall x \exists y P(x, y)$$
  $$\exists x \forall y P(x, y)$$
  $$\exists x \exists y P(x, y)$$

Quantifiers

- Summary:

	$\forall x P(x)$	$\exists x P(x)$
True	P(x) true for every x	P(x) true for a x (at least one x)
False	P(x) false for a x	P(x) false for every x

- Example: P(x): $x^2 < 10$, for $0 < x < 4$

  $\forall x P(x)$

  $\exists x P(x)$

Negations of Quantification

- Equivalences involving the negation operator

$$\neg\forall x P(x) \Leftrightarrow \exists x \neg P(x)$$

$$\neg\exists x P(x) \Leftrightarrow \forall x \neg P(x)$$

  - Distributing a negation operator across a quantifier changes a universal to an existential and vice versa.
  - Multiple Quantifiers: read left to right . . .
- Example:
  - P(x): x has taken MATH101, x $\in$ a student in this class.

Negations of Quantification

- Show that:

$$\neg \forall x (x^2 > x) \Leftrightarrow \exists x (x^2 \leq x)$$

Converting from English

- Steps involved:
  - Define predicates
  - Search for logical operators
  - Add quantifiers

Converting from English

- Example 1:

If a student is in this class, then the student has studied algebra.

S(x): x has studied algebra

C(x): x is in this class

Universe of discourse: all students.

Converting from English

- Example 2:

If a person is female and is parent, then this person is someone’s mother.

F(x): x is female

P(x): x is a parent

M(x, y): x is the mother of y

Converting from English

- Example 3:

I(x): x has a internet connection

C(x, y): x, y have chatted over the internet

x, y : a student in our class

Universe of discourse: all students in our class.

1). No one in the class has chatted with Bob.

2). Mike has chatted with everyone except Joseph.

3). Someone in your class has an Internet connection but has not chatted with anyone else in our class.

Converting to English

- Example 1:

$\forall x\ (C(x) \vee \exists y\ (C(y) \wedge F(x, y))\ )$

x, y: students in our school

C(x): x has a computer

F(x, y): x and y are friends

Converting to English

- Example 2:

C(x, y): x is enrolled in y

x: a member of the set of all students

y: a member of the set of all classes

1). C(David, ENGR213)

2). $\exists y\ C(\text{Mike}, y)$

3). $\exists x\ (\ C(x, \text{ENGR213}) \land C(x, \text{ENGR211})\ )$

4). $\exists x\ \exists y\ \forall z\ (\ (x \neq y) \land (C(x, z) \to C(y, z))\ )$

x, y: students

z: a course