Logic, Sets

Section 1.7 Set Operations

Logic and Set Theory

Propositional calculus and set theory are both instances of an algebraic system called a

Boolean Algebra

The operators in set theory are defined in terms of the corresponding operators in propositional calculus

Set Equivalence

- **Definition:** Two sets A and B are *equal*, denoted $A = B$, iff $\forall x[x \in A \leftrightarrow x \in B]$.

Note: By a previous logical equivalence we have

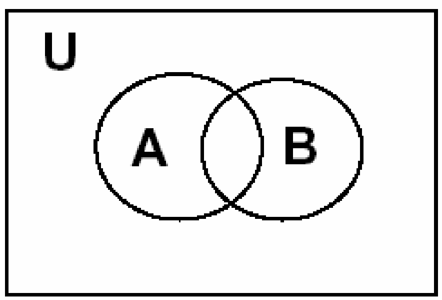
$$\mathrm{A} = \mathrm{B} \text{ iff } \forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

Or

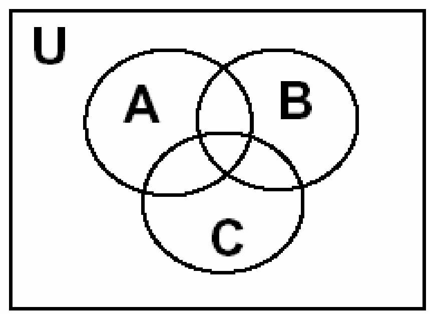
$$\mathrm{A} = \mathrm{B} \text{ iff } A \subseteq B \text{ and } B \subseteq A$$

Venn Diagrams

- A useful visualization tool (for 3 or less sets)
  - The Universe U is the rectangular box
  - Each set is represented by a circle and its interior
  - All possible combinations of the sets must be represented



For 2 sets



For 3 sets

Set Operations

- The ***union*** of A and B, denoted $A \cup B$, is the set $\{x \mid x \in A \vee x \in B\}$
- The ***intersection*** of A and B, denoted $A \cap B$, is the set $\{x \mid x \in A \wedge x \in B\}$
  Note: If the intersection is void, A and B are said to be *disjoint.*
- The ***complement*** of A, denoted $\overline{A}$, is the set $\{x \mid \neg(x \in A)\}$

Set Operations

- The ***difference*** of A and B, or the *complement* of B *relative to* A, denoted A - B, is the set

  $A \cap \overline{B}$

  Note: The (absolute) complement of A is U - A.
- The ***symmetric difference*** of A and B, denoted $A \oplus B$, is the set

  $(A - B) \cup (B - A)$

Venn Diagram for Set Operations

- Confirming Identities with Venn Diagrams: shade the appropriate region to represent the given set operation.
- Example 1: $\cap$ distributes over $\cup$.
  $A \cap (B \cup C)=(A\cap B) \cup (A\cap C)$
- Example 2: $\cup$ distributes over $\cap$
  $A \cup (B\cap C)=(A\cup B) \cap(A\cup C)$

Set Identities

- Set identities correspond to the logical equivalences.

Identity Law:

$A \cup \varnothing = A$ $\quad$ $A \cap U = A$

Domination Law:

$A \cup U = U$ $\quad$ $A \cap \varnothing = \varnothing$

Idempotent Law:

$A \cup A = A$

$A \cap A = A$

- Complementation Law:

$\overline{\overline{A}} = A$

Set Identities

Commutative Law:

$A \cup B = B \cup A$

Associative Law:

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

Distributive Law:

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan' Law:

$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Union and Intersection of Indexed Collections

- Let A_1, A_2 , ..., A_n be an indexed collection of sets.
  Union and intersection are associative (because 'and' and 'or' are) we have:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

- Examples: Let $A_i = [i, \infty),\ 1 \le i < \infty$

$$\bigcup_{i=1}^{n} A_i = ? \qquad \bigcap_{i=1}^{n} A_i = ?$$

Membership Tables

- Used to indicate whether an element in a set using 1 (in the set) or 0 (not in the set)
- If two sets have the same elements in a membership table, then they are equal.
- Example: De Morgan's Law

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$