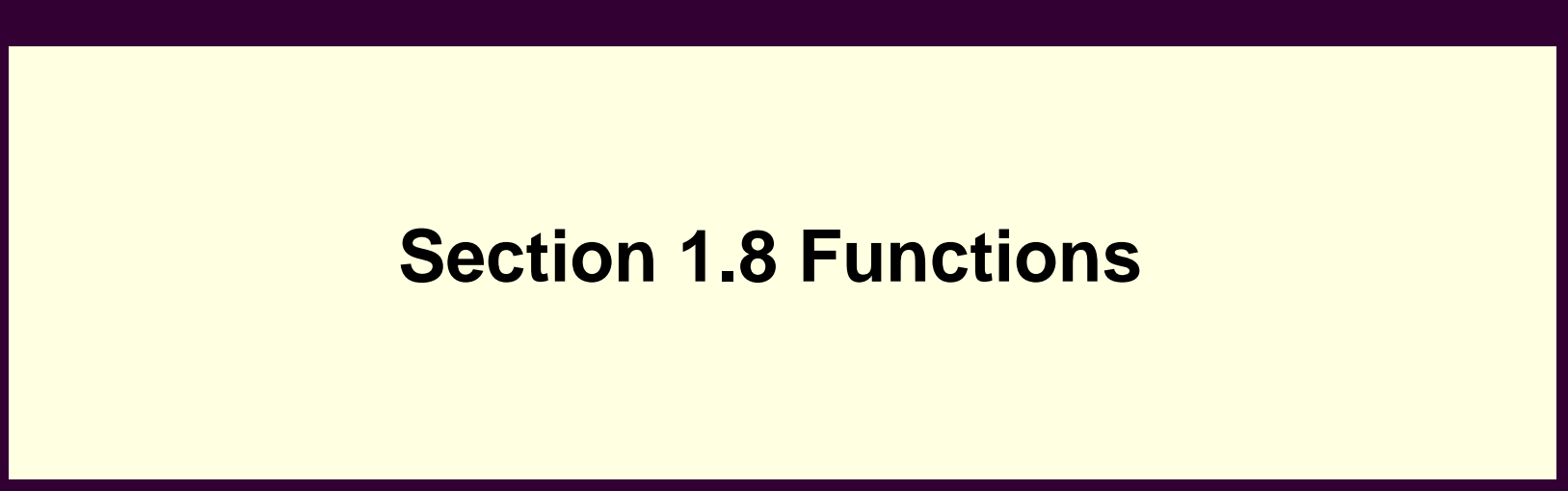




Functions, Complexity of Algorithms



Section 1.8 Functions

Definitions

- **Function:** Let A and B be sets. A function (mapping, map) f from A to B , denoted $f: A \rightarrow B$, is a subset of $A \times B$, and is a rule that assigns to each element $a \in A$ exactly one element $f(a) \in B$, called the value of f at a .
 - f associates with each x in A one and only one y in B
- We also say that $f: A \rightarrow B$ is a mapping from domain A to codomain B .
 - A is called the ***domain*** and
 - B is called the ***codomain***.

Definitions

- If $f(x) = y$
 - y is called the **image** of x under f
 - x is called a **preimage** of y

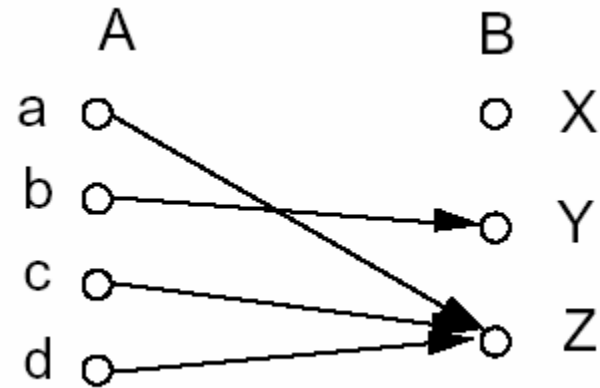
Note there may be more than one preimage of y but there is only one image of x .

- The **range** of f is the set of all images of points in A under f . We denote it by $f(A)$.
- If S is a subset of A then $f(S) = \{f(s) \mid s \text{ in } S\}$.

Definitions

■ Example 1:

- $f(a) = Z, \dots$
- the image of d is Z
- the preimage of Y is b
- the preimages of Z are a, c and d
- the domain of f is $A = \{a, b, c, d\}$
- the codomain is $B = \{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$
- $f(A) = \{Y, Z\}$
- $f(\{c,d\}) = \{\mathbf{Z}\}$



Definitions

- Example 2: Sequence of functions from \mathbb{R} to \mathbb{R} . Falling powers:

$$x^{\underline{n}} = x(x - 1) \cdots (x - n + 1).$$

- Example 3: C compiler maps the set of ASCII strings to the boolean set.

Special Functions

- The ***floor function***, denoted $f(x) = \lfloor x \rfloor$ or $f(x) = \text{floor}(x)$, is the largest integer less than or equal to x .
 - Note: the floor function is equivalent to truncation for positive numbers.
- The ***ceiling function***, denoted $f(x) = \lceil x \rceil$ or $f(x) = \text{ceiling}(x)$, is the smallest integer greater than or equal to x .
 - Note: the ceiling function is equivalent to truncation for negative numbers.

Properties of Functions

Let f be a function from A to B .

- **Definition:** f is *one-to-one* (denoted 1-1) or *injective* if and only if $f(x_1)=f(x_2)$ imply $x_1=x_2$.

Note: this means that if $a \neq b$ then $f(a) \neq f(b)$.

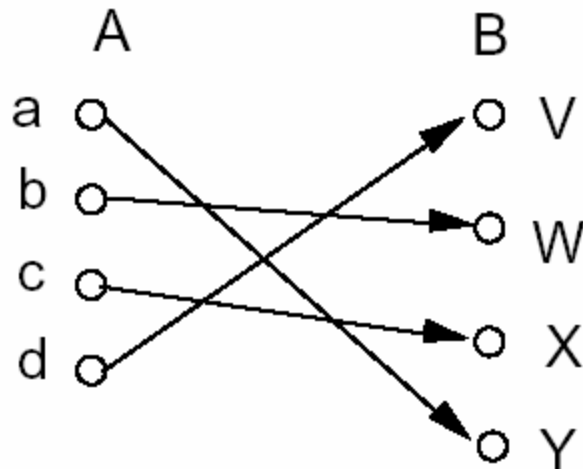
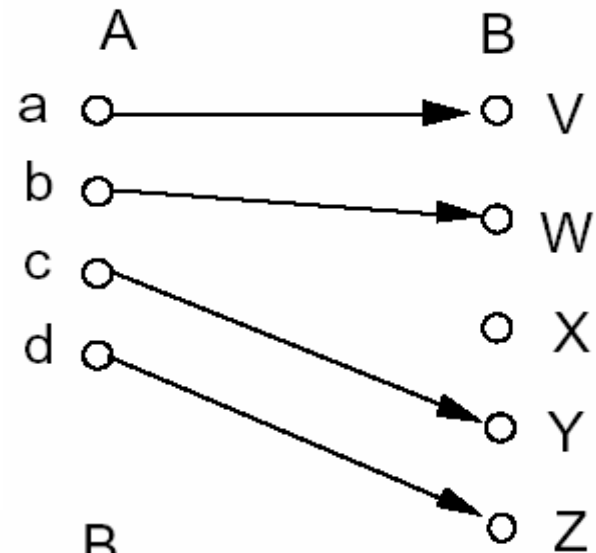
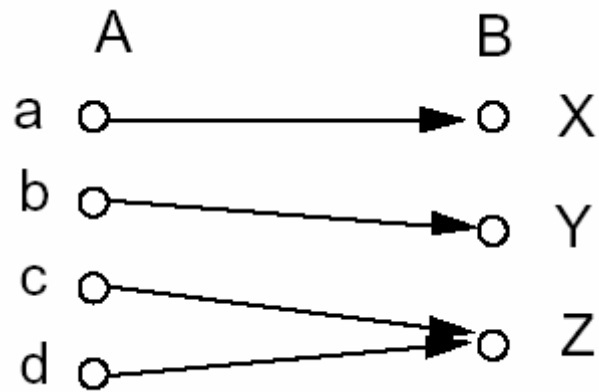
- **Definition:** f is *onto* or *surjective* if every y in B has a preimage.

Note: this means that for every y in B there must be an x in A such that $f(x) = y$.

- **Definition:** f is *bijective* if it is surjective and injective (one-to-one and onto).

Properties of Functions

■ Example: three functions



Properties of Functions

- One-to-one: horizontal line test.
Onto: scope test
- Whenever there is a bijection from A to B , the two sets must have the same number of elements or the same *cardinality*.

Properties of Functions

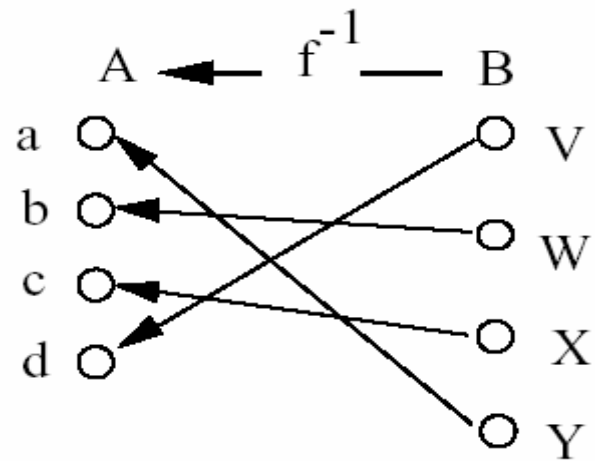
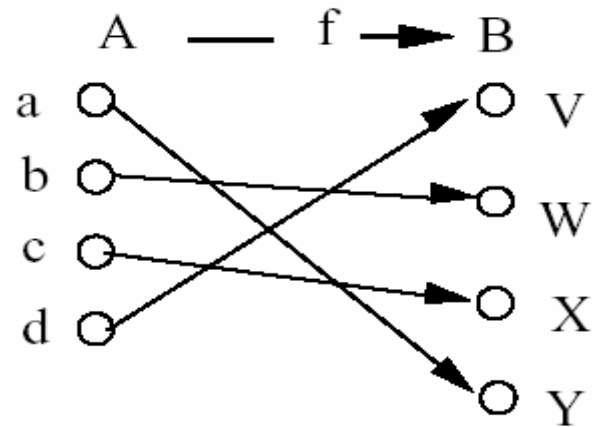
- *Examples: Let $A = B = \mathbb{R}$, the reals. Determine which are injections, surjections, bijections:*
 - $f(x) = x$
 - $f(x) = x^2$
 - $f(x) = x^3$
 - $f(x) = |x|$

Inverse Functions

- **Definition:** Let f be a bijection from A to B . Then the *inverse* of f , denoted f^{-1} , is the function from B to A defined as
$$f^{-1}(y) = x \text{ iff } f(x) = y$$
- **Note:** No inverse exists unless f is a bijection.

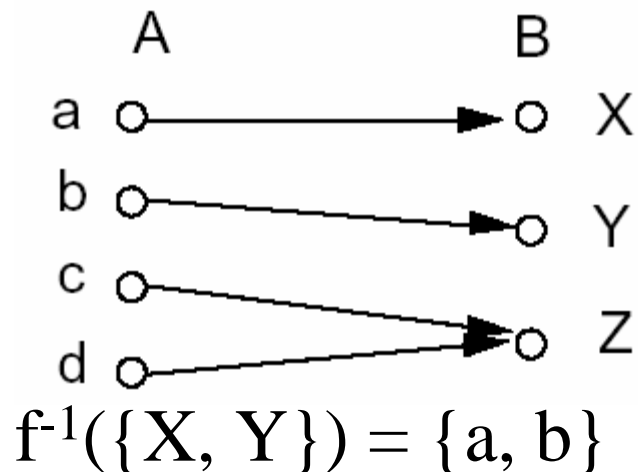
Inverse Functions

- Example: Let f be defined by the diagram:



Inverse Functions

- **Definition:** Let S be a subset of B . Then $f^{-1}(S) = \{x \mid f(x) \in S\}$
Note: f need not be a bijection for this definition to hold.
- **Example:** Let f be the following function:

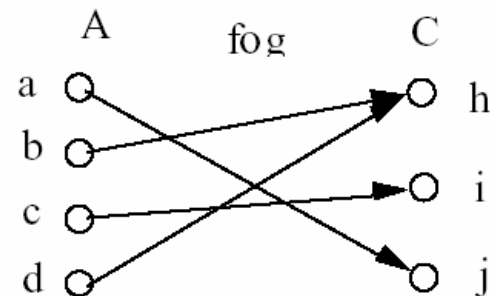
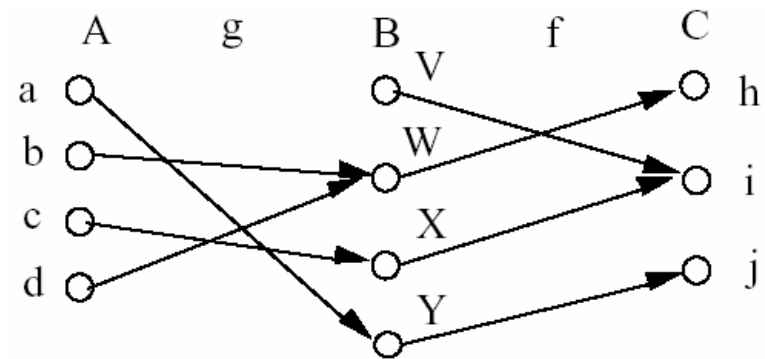


Composition

- **Definition:** Let $f: B \rightarrow C$, $g: A \rightarrow B$. The **composition** of f with g , denoted $f \circ g$, is the function from A to C defined by

$$f \circ g(x) = f(g(x))$$

- **Example:**



Composition

- Examples:

$$f(x) = x^2 \text{ and } g(x) = 2x + 1$$

$$f(g(x)) = ?$$

$$g(f(x)) = ?$$