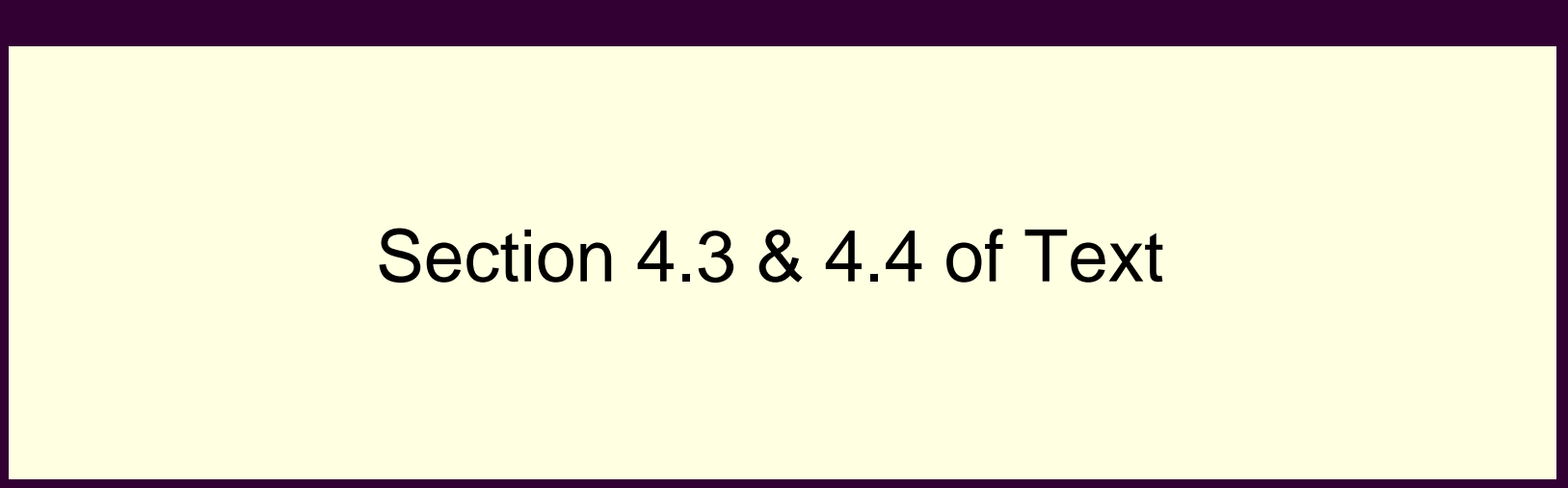




# Permutations, Combinations and Binomial Theorem



Section 4.3 & 4.4 of Text

# Permutation - Definition

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- **Definition:** **Permutation** is an ordered arrangement of distinct objects of a set.
  - A permutation is an arrangement that order matters.
  - After selecting the objects, two different orderings or arrangements constitute different permutations.
- **Notation:** An ordered  $r$ -selection from a set  $S$  of  $n$  elements (“traditionally” named:  $r$ -permutation) is a sequence of  $r$  objects from  $S$ , denoted as:  $P(n, r)$ .

# Permutation - Calculation

- Example: passwords of length 6 from distinct alphabets.
- In general, to compute r-permutation of a set with n element  $P(n, r)$ :
  - - Choose the first object n ways,
  - - Choose the second object (since selection is without replacement)  $(n - 1)$  ways,
  - - ...
  - - the rth object  $(n - r + 1)$  ways.

Thus,  $P(n, r) = n(n - 1)(n - 2) \dots (n - r + 1)$

Note:

$$P(n, r) = \frac{n!}{(n - r)!}$$

# Permutation - Examples

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- Suppose a salesman has to visit 8 different cities. He has to begin his trip in a specified city and end his trip in a specified city. He can visit the other cities in any order he wishes. How many possible orders can he use?
- How many different ways can three of the letters of the word BYTES be chosen and written in a row? How many different ways can this be done if the first letter must be B?

# Combinations - Definition

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- Definition: an **r-combination** of elements of a set of  $n$  elements is an unordered selection of  $r$  distinct elements from the set. It is simply a subset of the set with  $r$  elements.
- It is equivalent to selecting subsets of size  $r$  from a set of size  $n$ , denoted as  $C(n, r)$ .

# Combinations - Calculation

- Example: how many unordered 2 selection of elements can be made from the set  $\{0, 1, 2, 3\}$ .
- In general, the  $r$  permutation  $P(n, r)$  can be obtain by:
  - Form the  $r$ -combinations  $C(n, r)$ .
  - Ordering the  $r$  elements within each  $r$ -combination set, which can be done in  $P(r, r)$  way.

Thus,  $P(n, r) = C(n, r)P(r, r)$ , and

$$C(n, r) = \binom{n}{r} = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)!r!}$$

# Combinations - Examples

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- How many ways are there choosing 3 members from a group of 12 seniors to form a team for the ACM programming contest.
- How many ways are there to select a committee to develop a discrete mathematics course if the committee is to consist of 3 faculty from the math department and 4 from computer department, if there are 9 faculty members in math department and 11 members in computer department.

# Permutation vs. Combination

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- Two distinct methods to select  $r$  elements from a set of  $n$  elements.
- In an ordered selection (permutation), it is not only what elements are chosen but also the order in which they are chosen.
- In an unordered selection (combination), it is only the identity of the chosen elements that matters.

# Exercises

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- The IEEE club has 15 members
  - How many ways are there to choose four members of the club to serve on an executive committee?
  - How many ways to choose a president, vice president, secretary and treasurer of the club?
- There are 6 candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?
- A circuit consists of 10 components; 2 of them are faulty. How many different sets of locations for these two faulty components are possible?

# Combination - Calculation

- Theorem: the number of r-combination of a set with n elements is

$$C(n, r) = \binom{n}{r} = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)!r!}$$

- Interesting facts about some combination values:
  - $C(n, 0)$
  - $C(n, 1)$
  - $C(n, n)$

# More on Combination

- **Corollary:**  $C(n, r) = C(n, n-r)$ 
  - proof; why?
- **Corollary:**  $\sum_{r=0}^n C(n, r) = 2^n$
- **Corollary:** Pascal's Identity:  
 $C(n+1, k) = C(n, k-1) + C(n, k)$   
It produces Pascal's triangle.

# Example

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- Suppose you flip a fair coin  $n$  times. How many different ways can you get
  - no heads?
  - exactly one head?
  - exactly two heads?
  - exactly  $r$  heads?
  - ...
  - at least 2 heads?

# Example

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- How many bit string (consisting of 0s and 1s) of length 4 have exactly 2 ones (or exactly 2 zeros)?

# Example

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- How many bit string of length 100 have at least 2 ones?

# The Binomial Theorem

- Example: expansion of  $(x+y)^2$ ,  $(x+y)^3$

- The Binomial Theorem:  $n$  nonnegative integer,

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \dots + C(n, n-1)xy^{n-1} + C(n,n)y^n$$

or:

$$(x+y)^n = \sum_{j=0}^n C(n, j)x^{n-j}y^j$$

- Example: What is the coefficient of  $x^{25}y^{75}$  in the expansion of  $(2x-5y)^{100}$