

Solving Recurrence Relations

(Section 6.2 of the Text)

Review

- nth degree polynomials have n roots

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

- If the coefficients are real then the roots are real or occur in complex conjugate pairs.

- Recall the *quadratic formula*:

$$ax^2 + bx + c = 0 \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Review

- We assume you remember how to solve linear systems
 $Ax = b$.
where A is an $n \times n$ matrix.
- Example:

Statement of the Problem

- If $a_{g(n)} = f(a_{g(0)}, a_{g(1)}, \dots, a_{g(n-1)})$ find a closed form or an expression for $a_{g(n)}$.
- Example:

$$f(n) = 3^n \leftrightarrow f(0) = 1, f(n) = 3f(n-1)$$

Form of Recurrence Equations

- Solving recurrence relations can be very difficult unless the recurrence equation has a special form:
 - $g(n) = n$ (single variable)
 - the equation is *linear*:
 - sum of previous terms
 - no transcendental functions of the a_i 's
 - no products of the a_i 's
 - *constant* coefficients: the coefficients in the sum of the a_i 's are constants, independent of n .

Form of Recurrence Equations (cont.)

- unless the recurrence equation has a special form
 - *degree k*: a_n is a function of only the previous k terms in the sequence
 - *homogeneous*: If we put all the a_i 's on one side of the equation and everything else on the right side, then the right side is 0. Otherwise *inhomogeneous* or *nonhomogeneous*.

Form of Recurrence Equations (cont.)

A recurrence relation

$$a_n = f(a_{n-1}, \dots, a_{n-k})$$

has **degree k** if the function f depends on the term a_{n-k} and if it depends on no terms of lower index. It is **linear** of degree k if it has the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + g(n)$$

where each c_k is a real function and $c_k \neq 0$. It is **homogenous** if $g(n) = 0$.

Examples

	linear	const. coef.	homo- geneous	degree
$a_n = (1.02)a_{n-1}$				
$a_n = (1.02)a_{n-1} + 2^{n-1}$				
$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$				
$a_n = ca_{n/m} + b$				
$a_n = na_{n-1} + n^2a_{n-2} + a_{n-1}a_{n-2}$				

Solving Recurrence Equations

- A linear homogeneous recurrence of degree k with constant coefficients (“ k -LiHoReCoCo”) is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + \dots + c_k a_{n-k},$$

where the c_i are all real, and $c_k \neq 0$.

- The solution is uniquely determined if k initial conditions $a_0 \dots a_{k-1}$ are provided.

Solution Procedure

- Equation: k -LiHoReCoCo

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

- **Steps:**

1. Put all a_i 's on the left side of the equation, everything else on the right. If nonhomogeneous, stop (for now).

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = 0$$

2. Assume a solution of the form $a_n = b^n$. Substitute the solution into the equation, factor out the lowest power of b and eliminate it.

$$b^n - c_1 b^{n-1} - c_2 b^{n-2} - \cdots - c_k b^{n-k} = 0$$

$$b^{n-k} [b^k - c_1 b^{k-1} - \cdots - c_k] = 0$$

Solution Procedure (cont.)

- Steps

3. The remaining polynomial of degree k ,

$$b^k - c_1 b^{k-1} - \dots - c_k$$

is called the *characteristic polynomial*. The corresponding equation is called *characteristic equation*.

Find its k roots, r_1, r_2, \dots, r_k .

4. If the roots are distinct, the *general solution* is:

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n$$

Solution Procedure (cont.)

- Steps

5. The coefficients a_1, a_2, \dots, a_k are found by enforcing the initial conditions.

Solve the resulting linear system of equations:

$$a_0 = \alpha_1 r_1^0 + \alpha_2 r_2^0 + \dots + \alpha_k r_k^0$$

$$a_1 = \alpha_1 r_1^1 + \alpha_2 r_2^1 + \dots + \alpha_k r_k^1$$

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$$a_{k-1} = \alpha_1 r_1^{k-1} + \alpha_2 r_2^{k-1} + \dots + \alpha_k r_k^{k-1}$$

Examples

- Solving the recurrence equation:
 $a_n = 3a_{n-1}, a_0 = 4$
- Solution:
 - Step 1: Bring subscripted variables to one side: $a_n - 3a_{n-1} = 0$.
 - Step 2: Substitute $a_n = b^n$ and factor lowest power of b :
 $b^{n-1}(b-3) = 0$ or $b-3 = 0$
 - Step 3: Find the root of the characteristic equation: $r_1 = 3$

Examples (cont.)

- Solution (cont.)

- Step 4: Compute the general solution:

$$a_n = c3^n$$

- Step 5: Find the constants based on the initial conditions:

$$a_0 = c(3^0) \text{ or } c = 4$$

- Finally: Produce the specific solution:

$$a_n = 4(3^n)$$

Examples

- Solving the recurrence equation:

$$a_n = 3a_{n-2}, a_0 = a_1 = 1$$

The Case of Degenerate Roots

- If a root r_1 has multiplicity p , then the solution corresponding the root is:
 $(\alpha_1 + \alpha_2 n + \dots + \alpha_p n^{p-1}) r_1^n$
- For quadratic case: if the characteristic equation $r^2 - c_1 r - c_2 = 0$ has only 1 root r_0 , then,
 $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for all $n \geq 0$,
for some constants α_1, α_2 .

Examples

- Solving the recurrence equation:

$$a_n = 6a_{n-1} - 9a_{n-2}, a_0 = a_1 = 1$$

General Case of Repeated and Distinct Roots

- If roots r_1, r_2, \dots, r_t has multiplicities of p_1, p_2, \dots, p_t , then the general solution has the format:

$$\begin{aligned} & (\alpha_{11} + \alpha_{12}n + \dots + \alpha_{1p_1}n^{p_1-1})r_1^n + \\ & (\alpha_{21} + \alpha_{22}n + \dots + \alpha_{2p_2}n^{p_2-1})r_2^n + \dots + \\ & (\alpha_{t1} + \alpha_{t2}n + \dots + \alpha_{tp_t}n^{p_t-1})r_t^n \end{aligned}$$