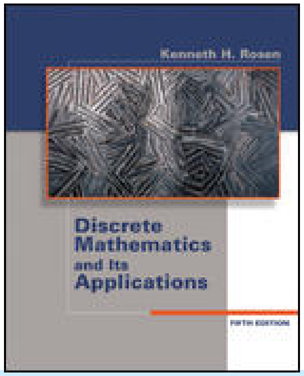
# Chapter 8 (Part 2): Graphs

- Representing Graphs & Graph Isomorphism (8.3)
- Connectivity (8.4)



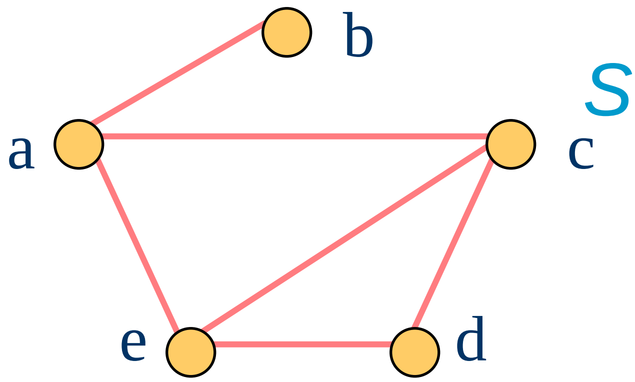
Partly adapted from the notes of Dr. D. Bouchaffra of Oakland University and Dr. M Frank of University of Florida

# Representing Graphs & Graph Isomorphism

- Introduction: goals
  - Choose the most convenient representation of a graph
  - We need to determine whether 2 graphs are isomorphic, this problem is important in graph theory.

# Representing Graphs - Adjacency Lists

- Use adjacency list, which specifies the vertices that are adjacent to each vertex of the graph
  - Example: Use adjacency lists to describe this simple graph.

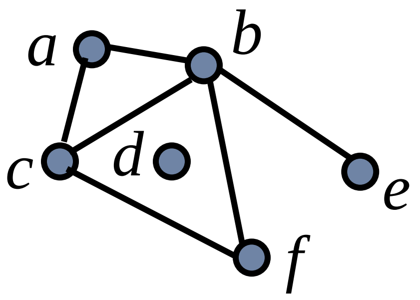


*Solution:*

| Vertex | Adjacent vertices |
|---|---|
| a | b, c, e |
| b | a |
| c | a, d, e |
| d | c, e |
| e | a, c, d |

# Adjacency Lists (cont)

- Example: Adjacency list



| *Vertex* | *Adjacent Vertices* |
|---|---|
| *a* | *b, c* |
| *b* | *a, c, e, f* |
| *c* | *a, b, f* |
| *d* | |
| *e* | *b* |
| *f* | *c, b* |

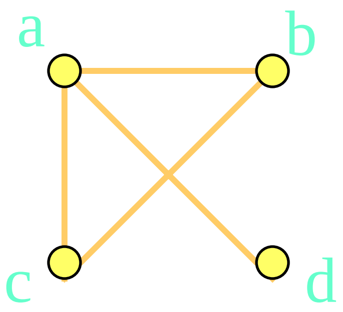
# Representing Graphs – Adjacency Matrices

- Adjacency matrices
  - – To simplify computation, graphs can be represented using matrices
  - – The adjacency matrix is defined as A = $[a_{ij}]$ such that

$$a_{ij} = \begin{cases} 1 \text{ if } \{v_i, v_j\} \text{ is an edge of } G \\ 0 \text{ otherwise} \end{cases}$$

# Adjacency Matrices (cont.)

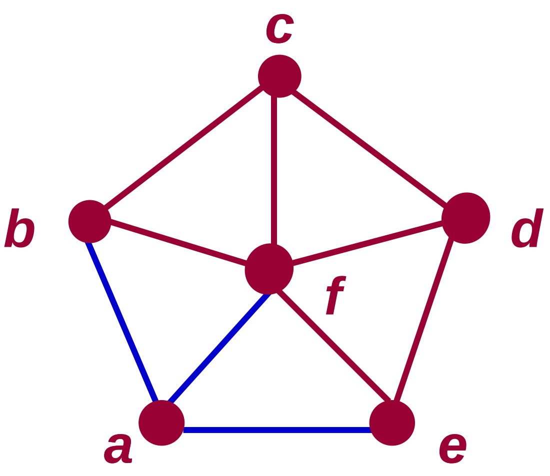
– Example: Use a adjacency matrix to represent this graph:



*Solution*: We order the vertices a, b, c, d. The matrix representing this graph is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

# Adjacency Matrices (Cont.)



$W_5$

**TO**

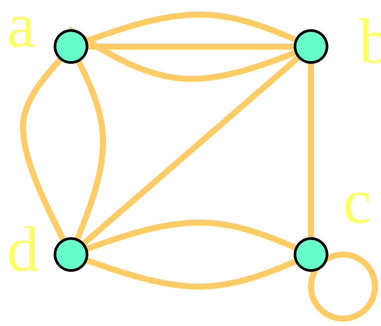
**FROM**

$$\begin{array}{cc} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \left[\begin{array}{cccccc} 0 & 1 & 0 & 0 & 1 & 1 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{array}\right] \end{array}$$

# Adjacency Matrices (cont.)

– In case of pseudographs , the adjacency matrix is not a binary matrix but is formed of elements that represent the number of edges between 2 vertices

– Example: Use an adjacency matrix to represent this pseudograph:



*Solution:* The adjacency matrix using The ordering of vertices a, b, c, d.

# Graph Isomorphism

- Isomorphism of graphs
  - Goal: is it possible to draw 2 graphs in the same way?
  - In chemistry, different compounds can have the same molecular formula but can differ in structure
  - The graphs of these compounds may not be drawn in the same way
  - Graphs having the same structure share common properties

# Graph Isomorphism (cont.)

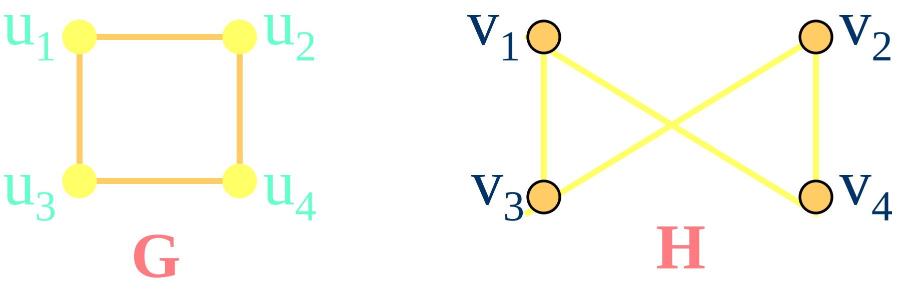
- Graph isomorphism:
  - Two graphs are isomorphic iff they are identical except for their node names.

- Definition 1:

  The simple graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are isomorphic if there is a one-to-one and onto function f from $V_1$ to $V_2$ with the property that a and b are adjacent in $G_1$ if and only if f(a) and f(b) are adjacent in $G_2$, for all a and b in $V_1$. Such a function f is called an isomorphism.

# Graph Isomorphism (cont.)

– Example: Show that the graphs G = (V,E) and H = (W,F) are isomorphic



*Solution:* The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$ is a one-to-one correspondence between V and W. To see that this correspondence preserves adjacency, note that adjacent vertices in G are $u_1$ and $u_2$, $u_1$ and $u_3$, $u_2$ and $u_4$, and $u_3$ and $u_4$, and each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, and $f(u_3) = v_3$ and $f(u_4) = v_2$ are adjacent in H.

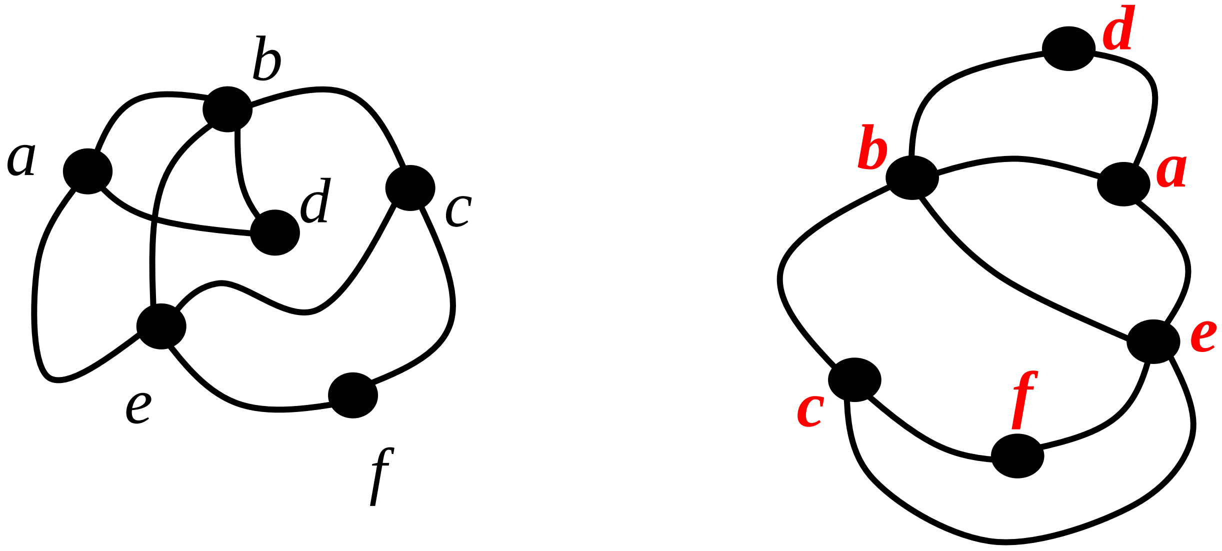
# Graph Invariants under Isomorphism

*Necessary* but not *sufficient* conditions for $G_1=(V_1, E_1)$ to be isomorphic to $G_2=(V_2, E_2)$:

- We must have that $|V1|=|V2|$, and $|E1|=|E2|$.
- The number of vertices with degree $n$ is the same in both graphs.
- For every proper subgraph $g$ of one graph, there is a proper subgraph of the other graph that is isomorphic to $g$.

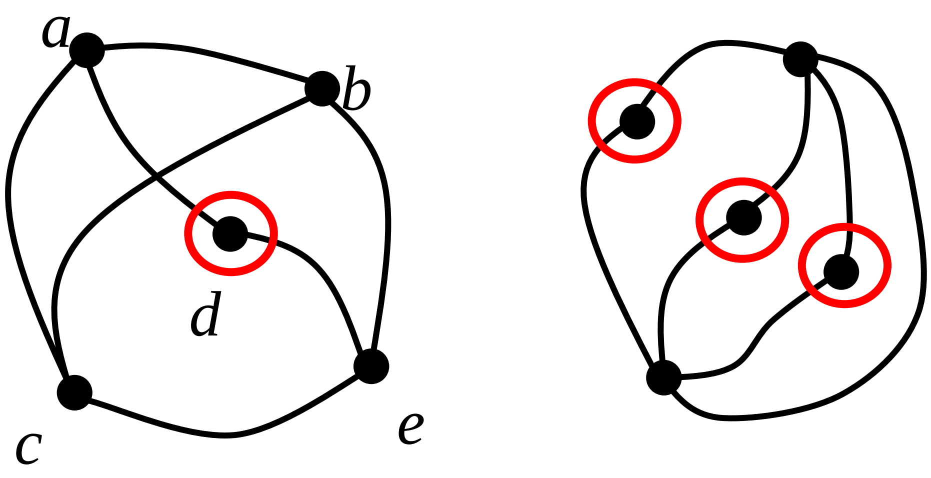
# Isomorphism Example

- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.



# Isomorphism Example

◆ If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.



- *# of vertices*
- *# of edges*
- *# of verts of the same degree*

# Connectivity (8.4)

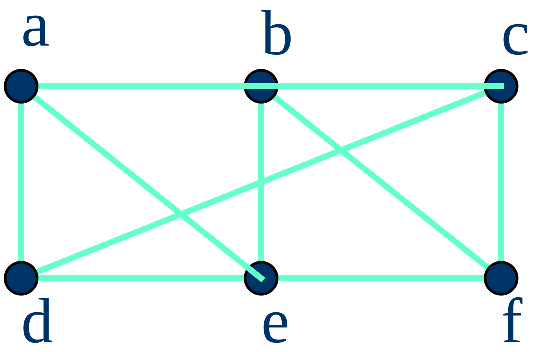
- Goal: determination of paths within graphs
- Many problems can be modeled with paths formed by traveling along the edges of graphs
- Some examples of problems are:
  - Study the link between remote computers
  - Efficient planning of routes for mail delivery
  - Garbage pickup
  - Diagnostic in computer networks

# Definitions

- In an undirected graph, a *path of length n from u to v* is a sequence of n adjacent edges going from vertex *u* to vertex *v*.
- A path is a *circuit* if $u=v$.
- A path *pass through* or *traverses* the vertices along it.
- A path is *simple* if it contains no edge more than once.

# A Example

- Example:
  - Path, simple path of length 4, circuit, simple path



# Paths in Directed Graphs

- Same as in undirected graphs, but the path must go in the direction of the arrows.

# Connectivity

- Connectedness in undirected graphs
  - Question asked:
    When does a computer network have the property that every pair of computers can share information, if message can be sent through one or more intermediate computers?
  - This question is equivalent to:
    When is there always a path between 2 vertices in the graph?

# Connectivity Definition

- Definition:

An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph

- Theorem:

There is a simple path between every pair of distinct vertices of a connected undirected graph

# Connectivity (cont.)

- A graph that is not connected is the union of two or more connected subgraphs, each pair of which has no vertex in common.

- These disjoint connected subgraphs are called the connected components of the graph

# Connectivity in Directed Graph

- Connected in directed graphs
  - Definition:

    A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph

  - Definition:

    A directed graph is weakly connected if there is a path between every 2 vertices in the underlying undirected graph

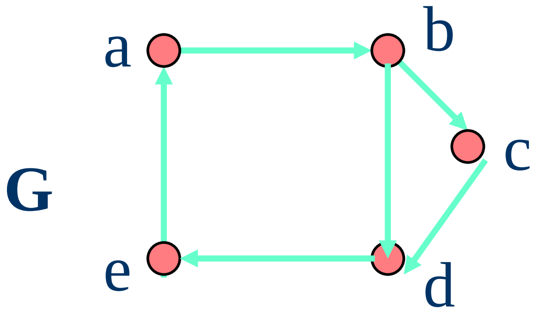
# Connectivity in Directed Graph (cont.)

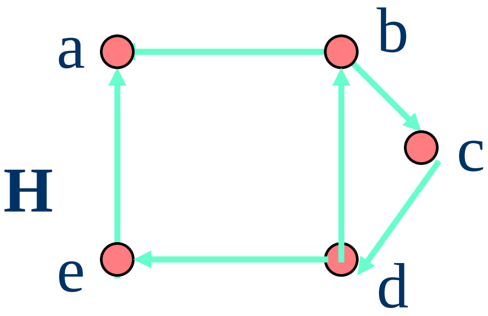
– A directed graph is weakly connected ⇔ there is always a path between 2 vertices when the directions of the edges are ignored

– Strongly connected ⇒ weakly connected directed graph

# Connectivity in Directed Graph (cont.)

- Example: Are the directed graphs G and H strongly connected?



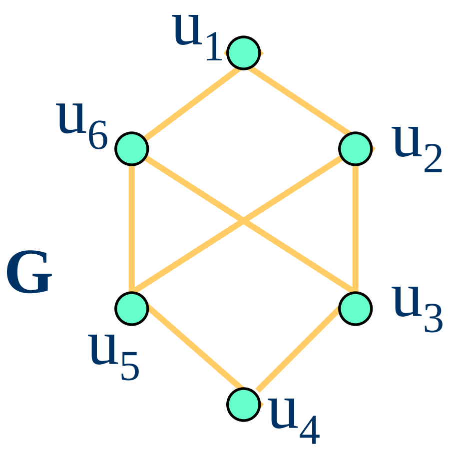


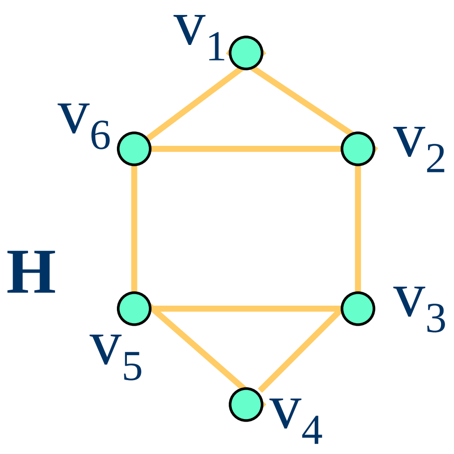
# Connectivity and Isomorphism

- Paths & isomorphism
  - Paths and circuits can help determine whether 2 graphs are isomorphic.
  - The existence of a simple circuit (or cycle) of a particular length is a useful invariant to show that 2 graphs are not isomorphic.

# Isomorphism Example

- Example: Determine whether the graph G and H are isomorphic.





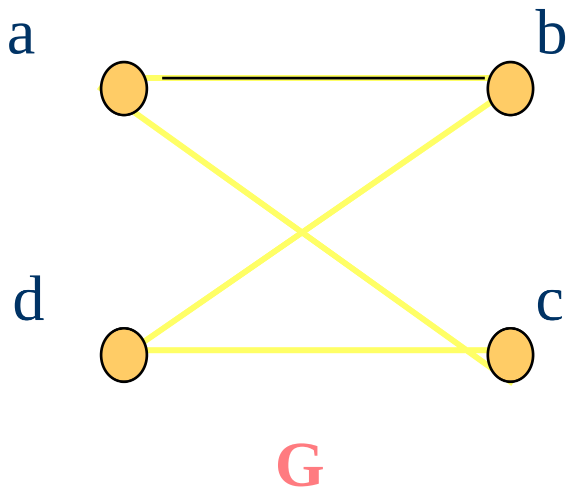
# Counting Paths

- Counting paths between vertices
  - Theorem:

Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$ (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from $v_i$ to $v_j$, where r is a positive integer is equals to the (i, j)th entry of $A^r$.

# Counting Paths (cont.)

– Example: How many paths of length 4 are there from a to d in the simple graph G?

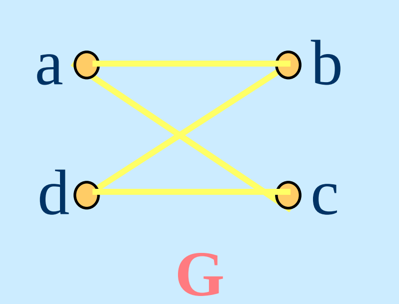


# Counting Paths (cont.)

Solution: The adjacency matrix of G (ordering the vertices as a, b, c, d) is

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Hence, the number of paths of length 4 from a to d is the (1,4)th entry of $A^4$. Since



$$A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}.$$