# Chapter 11

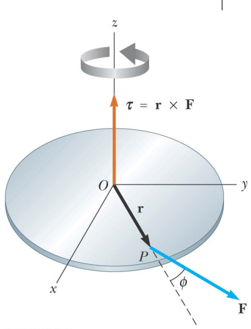
Angular Momentum

## The Vector Product

- There are instances where the product of two vectors is another vector
  - Earlier we saw where the product of two vectors was a scalar
    - This was called the dot product
- The vector product of two vectors is also called the cross product

## The Vector Product and Torque

- The torque vector lies in a direction perpendicular to the plane formed by the position vector and the force vector
- **τ** = **r** x **F**
- The torque is the vector (or cross) product of the position vector and the force vector



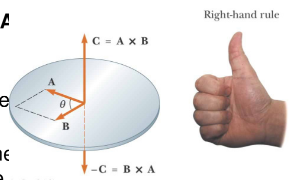



## The Vector Product Defined

- Given two vectors, **A** and **B**
- The vector (cross) product of **A** and **B** is defined as a third vector, **C**
- **C** is read as "**A** cross **B**"
- The magnitude of **C** is *AB* sin $\theta$
  - $\theta$ is the angle between **A** and **B**

## More About the Vector Product

- The quantity $AB \sin \theta$ is equal to the area of the parallelogram formed by **A** and **B**
- The direction of **C** is perpendicular to the plane formed by **A** and **B**
- The best way to determine this direction is to use the right-hand rule




## Properties of the Vector Product

- The vector product is *not* commutative. The order in which the vectors are multiplied is important
- To account for order, remember

  **A** x **B** = - **B** x **A**
- If **A** is parallel to **B** ($\theta = 0^{o}$ or $180^{o}$), then **A** x **B** = 0
  - Therefore **A** x **A** = 0

## Vector Products of Unit Vectors

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = -\hat{\mathbf{j}} \times \hat{\mathbf{i}} = \hat{\mathbf{k}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = -\hat{\mathbf{k}} \times \hat{\mathbf{j}} = \hat{\mathbf{i}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = -\hat{\mathbf{i}} \times \hat{\mathbf{k}} = \hat{\mathbf{j}}$$

## Vector Products of Unit Vectors, cont

- Signs are interchangeable in cross products
  - **A** x (-**B**) = - **A** x **B**
  -

$$\hat{\mathbf{i}} \times \left(-\hat{\mathbf{j}}\right) = -\hat{\mathbf{i}} \times \hat{\mathbf{j}}$$

## Using Determinants

- The cross product can be expressed as

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}\hat{\mathbf{k}}$$

- Expanding the determinants gives

$$\mathbf{A}\times\mathbf{B} = \left(A_yB_z - A_zB_y\right)\hat{\mathbf{i}} - \left(A_xB_z - A_zB_x\right)\hat{\mathbf{j}} + \left(A_xB_y - A_yB_x\right)\hat{\mathbf{k}}$$

## Torque Vector Example

- Given the force

$$\mathbf{F} = (2.00\,\hat{\mathbf{i}} + 3.00\,\hat{\mathbf{j}})\text{ N}$$
$$\mathbf{r} = (4.00\,\hat{\mathbf{i}} + 5.00\,\hat{\mathbf{j}})\text{ m}$$

$$\begin{aligned} \tau &= \mathbf{r} \times \mathbf{F} = [(4.00\hat{\mathbf{i}} + 5.00\hat{\mathbf{j}})\text{N}]\times[(2.00\hat{\mathbf{i}} + 3.00\hat{\mathbf{j}})\text{m}] \\ &= [(4.00)(2.00)\hat{\mathbf{i}}\times\hat{\mathbf{i}} + (4.00)(3.00)\hat{\mathbf{i}}\times\hat{\mathbf{j}} \\ &+ (5.00)(2.00)\hat{\mathbf{j}}\times\hat{\mathbf{i}} + (5.00)(3.00)\hat{\mathbf{i}}\times\hat{\mathbf{j}} \\ &= 2.0\,\hat{\mathbf{k}}\text{ N}\cdot\text{m} \end{aligned}$$

## Angular Momentum

- Consider a particle of mass *m* located at the vector position **r** and moving with linear momentum **p**

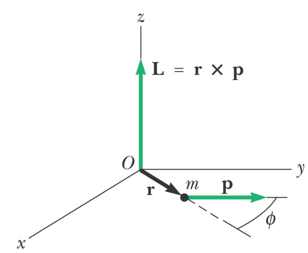
$$\mathbf{r}\times\sum\mathbf{F} = \sum\tau = \mathbf{r}\times\frac{d\mathbf{p}}{dt}$$

Adding the term $\frac{d\mathbf{r}}{dt}\times\mathbf{p}$

$$\sum\tau = \frac{d(\mathbf{r}\times\mathbf{p})}{dt}$$

## Angular Momentum, cont

- The instantaneous angular momentum **L** of a particle relative to the origin *O* is defined as the cross product of the particle's instantaneous position vector **r** and its instantaneous linear momentum **p**
  - **L** = **r** x **p**



## Torque and Angular Momentum

- The torque is related to the angular momentum
  - Similar to the way force is related to linear momentum

$$\sum \tau = \frac{d(\mathbf{r}\times\mathbf{p})}{dt} \qquad \sum \tau = \frac{d\mathbf{L}}{dt}$$

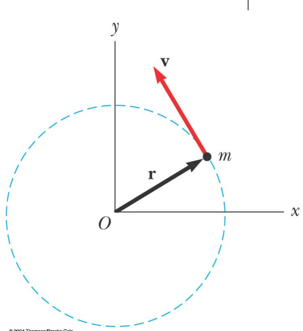
- This is the rotational analog of Newton's Second Law
  - $\sum \tau$ and **L** must be measured about the same origin
  - This is valid for any origin fixed in an inertial frame

## More About Angular Momentum

- The SI units of angular momentum are r x p = (kg·m$^2$)/ s
- Both the magnitude and direction of **L** depend on the choice of origin
- The magnitude of **L** = *mvr* sin $\phi$
  - $\phi$ is the angle between **p** and **r**
- The direction of **L** is perpendicular to the plane formed by **r** and **p**

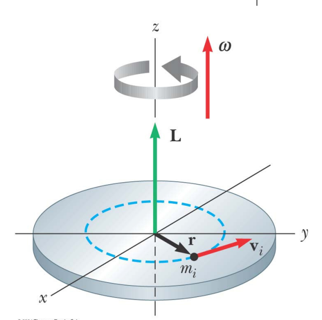
## Angular Momentum of a Particle, Example

- The vector **L** = **r** x **p** is pointed out of the diagram
- The magnitude is
  $L$ = *mvr* sin 90º = *mvr*
  - sin 90º is used since *v* is perpendicular to *r*
- A particle in uniform circular motion has a constant angular momentum about an axis through the center of its path




## Angular Momentum of a Rotating Rigid Object

- Each particle of the object rotates in the *xy* plane about the *z* axis with an angular speed of $\omega$
- The angular momentum of an individual particle is $L_i = m_i r_i^2 \omega$
- **L** and **ω** are directed along the *z* axis



## Angular Momentum of a Rotating Rigid Object, cont

- To find the angular momentum of the entire object, add the angular momenta of all the individual particles

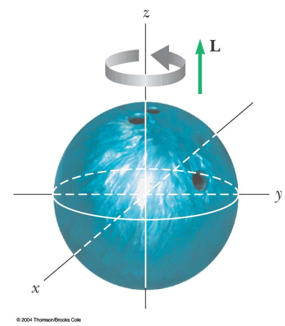
$$L_z = \sum_i L_i = \sum_i m_i r_i^2 \omega = I\omega$$

- This also gives the rotational form of Newton's Second Law

$$\sum \tau_{ext} = \frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha$$

## Angular Momentum of a Bowling Ball

- The momentum of inertia of the ball is $2/5MR^2$
- The angular momentum of the ball is $L_z = I\omega$
- The direction of the angular momentum is in the positive $z$ direction




## Conservation of Angular Momentum

- The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero
  - Net torque = 0 -> means that the system is isolated
- $\mathbf{L}_{tot}$ = constant or $\mathbf{L}_i = \mathbf{L}_f$
- For a system of particles, $\mathbf{L}_{tot} = \Sigma \mathbf{L}_n$ = constant

## Conservation of Angular Momentum, cont

- If the mass of an isolated system undergoes redistribution, the moment of inertia changes
  - The conservation of angular momentum requires a compensating change in the angular velocity
  - $I_i \omega_i = I_f \omega_f$
    - This holds for rotation about a fixed axis and for rotation about an axis through the center of mass of a moving system
    - The net torque must be zero in any case

## Conservation Law Summary

- For an isolated system -

(1) Conservation of Energy:

- $E_i = E_f$

(2) Conservation of Linear Momentum:

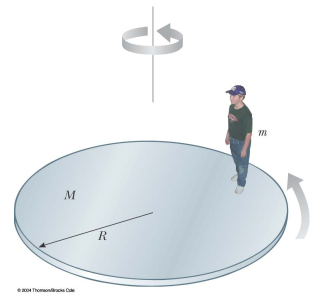
- $\mathbf{p}_i = \mathbf{p}_f$

(3) Conservation of Angular Momentum:

- $\mathbf{L}_i = \mathbf{L}_f$

## Conservation of Angular Momentum: The Merry-Go-Round

- The moment of inertia of the system is the moment of inertia of the platform plus the moment of inertia of the person
  - Assume the person can be treated as a particle
- As the person moves toward the center of the rotating platform, the angular speed will increase
  - To keep **L** constant




## Conservation of Angular Momentum: The Merry-Go-Round - Problem

A playground merry-go-round of radius $R$ = 1.40 m has a moment of inertia $I$ = 250 kg · m$^2$ and is rotating at 11.0 rev/min about a frictionless vertical axle. Facing the axle, a 25.0 kg child hops onto the merry-go-round and manages to sit down on its edge. What is the new angular speed of the merry-go-round? Answer: 9.2 rev/min